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J. Differential Equations 208 (2005) 388–429

Journal of Differential Equations

http://www.elsevier.com/locate/jde

Fredholm differential operators with unbounded coefficients ☆

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Received August 4, 2003

Abstract

We prove that a first-order linear differential operator **G** with unbounded operator coefficients is Fredholm on spaces of functions on $\mathbb{R}$ with values in a reflexive Banach space if and only if the corresponding strongly continuous evolution family has exponential dichotomies on both $\mathbb{R}_+$ and $\mathbb{R}_-$ and a pair of the ranges of the dichotomy projections is Fredholm, and that the Fredholm index of **G** is equal to the Fredholm index of the pair. The operator **G** is the generator of the evolution semigroup associated with the evolution family. In the case when the evolution family is the propagator of a well-posed differential equation $u'(t) = A(t)u(t)$ with, generally, unbounded operators $A(t), t \in \mathbb{R}$, the operator **G** is a closure of the operator $-\frac{d}{dt} + A(t)$. Thus, this paper provides a complete infinite-dimensional generalization of well-known finite-dimensional results by Palmer, and by Ben-Artzi and Gohberg.

MSC: 47D06; 35P05; 35F10; 58J20 (primary); 58E99; 47A53 (secondary)

Keywords: Exponential dichotomy; Fredholm operators; Fredholm index; Differential and difference operators; Evolution semigroups; Pairs of subspaces; Travelling waves; Spectral flow

☆This project has been funded in part by the National Academy of Sciences under the Collaboration in Basic Science and Engineering Program/Twinning Program supported by Contract No. INT-0002341 from the National Science Foundation. Also, the first author was supported in part by the Research Board and Research Council of the University of Missouri and by the CRDF grant UP1-2567-OD-03, and the second author was supported in part by the KBN grant 5 P03A 027 21. The authors thank K. Makarov and F. Gesztesy for many valuable discussions, and R. Schnaubelt for important comments on the preliminary version of the paper.

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0022-0396/$ - see front matter
doi:10.1016/j.jde.2003.10.018

1. Introduction and main results

The celebrated Dichotomy Theorem asserts that a $d \times d$-matrix linear differential operator

$$G = -\frac{d}{dt} + A(t), \tag{1.1}$$

acting on a space of d-dimensional vector-functions on $\mathbb{R}$, is Fredholm if and only if the differential equation $u'(t) = A(t)u(t),\ \ t \in \mathbb{R}$, has exponential dichotomies on both $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_- = (-\infty, 0]$; moreover, the Fredholm index of G is equal to the difference of the ranks of the dichotomies. Palmer proved this result in [36,37] for the case when G acts on a space of continuous vector-functions. Ben-Artzi and Gohberg [12] proved this result in the case when G acts on $L_2(\mathbb{R}; \mathbb{C}^d)$ and $A \in L_\infty(\mathbb{R}; \mathscr{L}(\mathbb{C}^d))$. Also, we remark on an earlier paper by Sacker [43], where the "if"-part of this result and the index formula were proved in the framework of linear skew-product flows over the hull of A. For further developments of the latter approach see [23,44,50], and the bibliographies therein.

The Dichotomy Theorem is important in many questions of finite-dimensional dynamics. This theorem is instrumental in the study of spectral stability of travelling waves; see, e.g. [45] and numerous references therein. Motivated by applications to the study of partial differential equations, several steps have been made to generalize the Dichotomy Theorem for infinite-dimensional setting and unbounded operators $A(t)$. We mention here important results in [20,21, Theorem 1.1; 28,31,32; 38, Theorem 1; 42,46, Theorem 2.6; 53] see also the bibliographies in these papers, and the work of Baskakov [4–9]. Also, recently the infinite-dimensional Dichotomy Theorem gained additional importance due to connections with infinite-dimensional Morse theory, see [2,3,20,41] and the literature therein. In the above-mentioned papers infinite-dimensional versions of the Dichotomy Theorem have been proved either for important particular classes of operators $A(t)$, or under some additional assumptions on the solutions of the differential equation $u' = A(t)u$ or its adjoint, or on the corresponding evolution family (the propagator of the differential equation). These assumptions have been used to deal with the following principal differences between the finite-dimensional and the infinite-dimensional settings: (a) Difficulties to prove the closedness of the subspaces of initial data that generate solutions of the equation $u'(t) = A(t)u(t)$ and, respectively, the adjoint equation, that are bounded at $+\infty$ and, respectively, $-\infty$ (see, e.g., [44, Lemma 7.6, 7.11(A); 38, Lemma 2.3]); (b) That the propagator of the differential equation or/and its adjoint may have a nontrivial kernel (see, e.g., [7, Assumption 1; 38, Hypothesis 5; 46, Hypothesis (U1)]); and (c) That both stable and unstable dichotomy subspaces for the equation might be infinite dimensional (cf. [25,38,44,46,51] and see Examples 7.1 and 7.2 below).

The main goal of the current paper is to prove an infinite-dimensional version of the Dichotomy Theorem without any special restrictions on the operators $A(t)$. The corresponding differential operator is considered on the space $L_p = L_p(\mathbb{R}; X)$,

$p \in [1, \infty)$, or on $C_0(\mathbb{R}; X)$, the space of continuous X-valued functions vanishing at $\pm\infty$. The Banach space X is assumed[1] to be reflexive. Both the formulation and the proof of the Dichotomy Theorem in this unrestricted setting are quite different from the ones known in the literature.

To achieve this goal, as our starting point, we consider not the differential equation $u' = A(t)u$, but a strongly continuous exponentially bounded evolution family $\{U(t, \tau)\}_{t \geqslant \tau}$, $t, \tau \in \mathbb{R}$, on X. In particular, if the differential equation is well-posed (see Section 7 and cf. [15, p. 58; 19, Definition VI.9.1]), then $U(t, \tau)$ is its propagator (Cauchy operator). A more important infinite dimensional issue is related to the definition of the operator G in (1.1). A quite natural first try is to define G, $(Gu)(t) = -u'(t) + A(t)u(t)$, say, on L_p, as an operator with the domain

$$\operatorname{dom} G = W_p^1 \cap \{u \in L_p : u(t) \in \operatorname{dom} A(t) \text{ a.e.}, \ A(\cdot)u(\cdot) \in L_p\}, \tag{1.2}$$

where $W_p^1 = W_p^1(\mathbb{R}; X)$, $p \in [1, \infty)$, is the Sobolev space so that $W_p^1 = \operatorname{dom}(-d/dt)$. This choice of G, however, appears to be unnecessarily restrictive since this operator might not be closed, see, e.g., [48, Section 2(c)]. To settle this issue, we consider instead a certain closed extension, $\mathbf{G}$, of the operator G. The operator $\mathbf{G}$ is the generator of a so called *evolution semigroup* $\{T^t\}_{t \geqslant 0}$ on L_p or $C_0(\mathbb{R}; X)$, see Lemma 1.3 below. Recently, the evolution semigroups and their generators have been successfully used to characterize stability of evolution families, and their exponential dichotomy on $\mathbb{R}$, see [15,48] and the bibliographies therein, [4,11,34,35], and also [17, Lemma IV.3.3; 27, Chapter 10] for a more classical but related approach. However, the complete characterization of the Fredholm property of the generator of the evolution semigroup given in this paper appears to be new. Our principal result reads as follows.

For $u \in C_0(\mathbb{R}; X)$, resp. $v \in C_0(\mathbb{R}; X^*)$, so that $u(t) = U(t, \tau)u(\tau)$, resp. $v(\tau) = U(t, \tau)^* v(t)$, for all $t \geqslant \tau$ in $\mathbb{R}$, we *assume* throughout that if $u(t) = 0$, resp. $v(t) = 0$, for some $t \in \mathbb{R}$ then $u = 0$, resp $v = 0$.

Theorem 1.1. *Assume that $\{U(t, \tau)\}_{t \geqslant \tau}$, $t, \tau \in \mathbb{R}$, is a strongly continuous exponentially bounded evolution family on a reflexive Banach space X, and let $\mathbf{G}$ denote the generator of the associated evolution semigroup defined on $L_p(\mathbb{R}; X)$, $p \in [1, \infty)$, or on $C_0(\mathbb{R}; X)$. Then*

$$\textit{the operator } \mathbf{G} \textit{ is Fredholm} \tag{1.3}$$

if and only if there exist $a \leqslant b$ in $\mathbb{R}$ such that the following two conditions hold:

(i) *The evolution family $\{U(t, \tau)\}_{t \geqslant \tau}$ has exponential dichotomies $\{P_t^-\}_{t \leqslant a}$ and $\{P_t^+\}_{t \geqslant b}$ on $(-\infty, a]$ and $[b, \infty)$, respectively.*

(ii) *The* node operator $N(b, a)$, *acting from* $\operatorname{Ker} P_a^-$ to $\operatorname{Ker} P_b^+$ *by the rule* $N(b, a) = (I - P_b^+)U(b, a)|_{\operatorname{Ker} P_a^-}$, *is Fredholm.*

[1] We suspect that one can remove the reflexivity assumption, mainly used in Proposition 3.4, but prefer not to pursue this here.

Moreover, if (1.3) *holds, then* $\dim \operatorname{Ker} \mathbf{G} = \dim \operatorname{Ker} N(b,a)$, $\operatorname{codim} \operatorname{Im} \mathbf{G} = \operatorname{codim} \operatorname{Im} N(b,a)$, *and* $\operatorname{ind} \mathbf{G} = \operatorname{ind} N(b,a)$.

Recall that a pair of subspaces (W,V) in X is called a *Fredholm pair* provided $\alpha(W,V) := \dim(W \cap V) < \infty$, the subspace $W+V$ is closed, and $\beta(W,V) := \operatorname{codim}(W+V) < \infty$; the *Fredholm index* of the pair is defined as $\operatorname{ind}(W,V) = \alpha(W,V) - \beta(W,V)$, see, e.g., [24, Section IV.4.1]. Theorem 1.1 can be equivalently reformulated as follows.

Theorem 1.2. *Under the assumptions in Theorem* 1.1, (1.3) *is fulfilled if and only if the following two conditions hold*:

(i′) *The evolution family* $\{U(t,\tau)\}_{t \geqslant \tau}$ *has exponential dichotomies* $\{P_t^-\}_{t \leqslant 0}$ *and* $\{P_t^+\}_{t \geqslant 0}$ *on* $\mathbb{R}_-$ *and* $\mathbb{R}_+$, *respectively.*

(ii′) *The pair of subspaces* $(\operatorname{Ker} P_0^-, \operatorname{Im} P_0^+)$ *is Fredholm in* X.

Moreover, if (1.3) *holds, then* $\dim \operatorname{Ker} \mathbf{G} = \alpha(\operatorname{Ker} P_0^-, \operatorname{Im} P_0^+)$, $\operatorname{codim} \operatorname{Im} \mathbf{G} = \beta(\operatorname{Ker} P_0^-, \operatorname{Im} P_0^+)$, *and* $\operatorname{ind} \mathbf{G} = \operatorname{ind}(\operatorname{Ker} P_0^-, \operatorname{Im} P_0^+)$.

Note that $N(0,0) = (I - P_0^+)|_{\operatorname{Ker} P_0^-} : \operatorname{Ker} P_0^- \to \operatorname{Ker} P_0^+$, and one can show that condition (ii′) in Theorem 1.2 is equivalent to condition (ii) in Theorem 1.1 with $a = 0 = b$, see Lemma 5.1 below.

Let J be one of the intervals $\mathbb{R}_+$, $\mathbb{R}_-$, or $\mathbb{R}$. Recall that a family $\{U(t,\tau)\}_{t \geqslant \tau}$, $t,\tau \in J$, of bounded linear operators on X is called a *strongly continuous exponentially bounded evolution family* on J if: (1) for each $x \in X$ the map $(t,\tau) \mapsto U(t,\tau)x$ is continuous for all $t \geqslant \tau$ in J; (2) for some $\omega \in \mathbb{R}$ the inequality $\sup\{\|e^{-\omega(t-\tau)}U(t,\tau)\| : t,\tau \in J, t \geqslant \tau\} < \infty$ holds; and (3) $U(t,t) = I$, $U(t,\tau) = U(t,s)U(s,\tau)$ for all $t \geqslant s \geqslant \tau$ in J. We say that $\{U(t,\tau)\}_{t \geqslant \tau}$ has *exponential dichotomy* $\{P_t\}_{t \in J}$ on J with *dichotomy constants* $M \geqslant 1$ and $\alpha > 0$ if P_t, $t \in J$, are bounded projections on X, and for all $t \geqslant \tau$ in J the following assertions hold:

(i) $U(t,\tau)P_\tau = P_t U(t,\tau)$ (intertwining, or invariance, property),

(ii) the restriction $U(t,\tau)|_{\operatorname{Ker} P_\tau}$ of the operator $U(t,\tau)$ is an invertible operator from $\operatorname{Ker} P_\tau$ to $\operatorname{Ker} P_t$;

(iii) the following *stable* and *unstable* dichotomy estimates hold:

$$\|U(t,\tau)|_{\operatorname{Im} P_\tau}\| \leqslant M e^{-\alpha(t-\tau)} \quad \text{and} \quad \|(U(t,\tau)|_{\operatorname{Ker} P_\tau})^{-1}\| \leqslant M e^{-\alpha(t-\tau)}.$$

For the notion of exponential dichotomy we refer to the classical books [22,27], and to newer work in [15,16,19,48,50], and the extensive bibliographies therein. Note that (i)–(iii) imply that for every $x \in X$ the function $t \to P_t x$ is continuous on J and $\sup_{t \in J} \|P_t\| < \infty$, see e.g. [35, Lemma 4.2] or [17, Lemma IV.1.1,IV.3.2].

Recall that the evolution semigroup $\{T^t\}_{t \geqslant 0}$ is defined on $L_p(\mathbb{R};X)$, $p \in [1,\infty)$, or on $C_0(\mathbb{R};X)$, by the formula $T^t u(\tau) = U(\tau,\tau-t)u(\tau-t)$, $\tau \in \mathbb{R}$, see [15]. This is a strongly continuous semigroup, and we let $\mathbf{G}$ denote its generator. Alternatively, the generator $\mathbf{G}$ can be described as follows (see [15, Proposition 4.32, 15, Lemma 3.16], and cf. [6, Theorem 1, 35, Lemma 1.1, 34, Lemma 1.1]).

Lemma 1.3. *A function u belongs to the domain* dom $\mathbf{G}$ *of the operator* $\mathbf{G}$ *on* $L_p(\mathbb{R};X)$, $p\in[1,\infty)$, *resp., on* $C_0(\mathbb{R};X)$, *if and only if* $u\in L_p(\mathbb{R};X)\cap C_0(\mathbb{R};X)$, *resp.*, $u\in C_0(\mathbb{R};X)$, *and there exists an* $f\in L_p(\mathbb{R};X)$, *resp.*, $f\in C_0(\mathbb{R};X)$, *such that*

$$u(t)=U(t,\tau)u(\tau)-\int_\tau^t U(t,s)f(s)\,ds,\quad \textit{for all } t\geqslant\tau \textit{ in } \mathbb{R}. \tag{1.4}$$

If (1.4) *holds, then* $\mathbf{G}u=f$.

We stress that (1.4) is a mild reformulation of the inhomogeneous differential equation $u'(t)=A(t)u(t)+f(t)$, $t\in\mathbb{R}$. If $\{U(t,\tau)\}_{t\geqslant\tau}$ is the propagator of a well-posed differential equation $u'(t)=A(t)u(t)$, $t\in\mathbb{R}$, with, generally, unbounded operators $A(t)$, then a subset of dom G from (1.2) is a core for $\mathbf{G}$, see [15, Theorem 3.12; 48, Proposition 4.1]. Thus, if the operator G with the domain dom G from (1.2) is a closed operator on $L_p(\mathbb{R};X)$, $p\in[1,\infty)$, resp., on $C_0(\mathbb{R};X)$, then $\mathbf{G}=G$.

Under an a priori assumption that assertion (i) in Theorem 1.1 holds, the equivalence of (1.3) and (ii), and the index formula, have been studied in [9, Theorem 4; 8, Theorem 8]. Therefore, in the current paper we will concentrate mostly on the main new contribution which is a proof of implication (1.3) $\Rightarrow$ (i′) in Theorem 1.2. Our strategy is to pass from the *differential* operator $\mathbf{G}$ on $L_p(\mathbb{R};X)$, $p\in[1,\infty)$, resp., on $C_0(\mathbb{R};X)$, to an associated *difference* operator, D, defined on the space $\ell_p(\mathbb{Z};X)$, $p\in[1,\infty)$, resp., on the space $c_0(\mathbb{Z};X)$ of sequences vanishing at $\pm\infty$, by the rule

$$D:(x_n)_{n\in\mathbb{Z}}\mapsto(x_n-U(n,n-1)x_{n-1})_{n\in\mathbb{Z}}. \tag{1.5}$$

This strategy has a long history that goes back to Henry [22, Theorem 7.6.5]. It was successfully used to treat the dichotomy on $\mathbb{R}$ and *invertible* operators $\mathbf{G}$, see [5, Theorem 2; 4, Theorem 2; 26, Lemma 3.3; 13, Section 5; 15, Theorem 4.16,4.37] (and also [15, Theorem 7.9]; [39, Theorem 4.1] or [50, Theorem 45.8] for a related case of linear skew-product flows on Banach spaces). The justification of this strategy for dichotomies on $\mathbb{R}_+$ and $\mathbb{R}_-$ and *Fredholm* operators $\mathbf{G}$ is given in the following theorem (cf. [7, Theorem 2; 6, Theorem 1; 52, Theorem 2]).

Theorem 1.4. *Assume that* $\{U(t,\tau)\}_{t\geqslant\tau}$, $t,\tau\in\mathbb{R}$, *is a strongly continuous exponentially bounded evolution family on a Banach space* X, *let* $\mathbf{G}$ *denote the generator of the associated evolution semigroup on* $L_p(\mathbb{R};X)$, $p\in[1,\infty)$, *resp.*, $C_0(\mathbb{R};X)$, *and let* D *be the difference operator on* $\ell_p(\mathbb{Z};X)$, $p\in[1,\infty)$, *resp., on* $c_0(\mathbb{Z};X)$, *defined in* (1.5). *Then* $\operatorname{Im}\mathbf{G}$ *is closed if and only if* $\operatorname{Im}D$ *is closed, and* $\dim\operatorname{Ker}\mathbf{G}=\dim\operatorname{Ker}D$ *and* $\operatorname{codim}\operatorname{Im}\mathbf{G}=\operatorname{codim}\operatorname{Im}D$. *In particular, the operator* $\mathbf{G}$ *is Fredholm if and only if* D *is Fredholm, and* $\operatorname{ind}\mathbf{G}=\operatorname{ind}D$.

By the following simple lemma, an exponential dichotomy on $\mathbb{Z}_\pm$ extends to an exponential dichotomy on $\mathbb{R}_\pm$ (cf. [22, Example 7.6.10]).

Lemma 1.5. *Assume that* $\{U(t,\tau)\}_{t\geqslant\tau}$, $t,\tau\in\mathbb{R}$, *is a strongly continuous exponentially bounded evolution family on a Banach space* X. *The discrete evolution family* $\{U(n,m)\}_{n\geqslant m}$, $n,m\in\mathbb{Z}$, *has an exponential dichotomy* $\{P_n^+\}_{n\geqslant 0}$ *on* $\mathbb{Z}_+$, *resp.*, $\{P_n^-\}_{n\leqslant 0}$ *on* $\mathbb{Z}_-$, *if and only if the family* $\{U(t,\tau)\}_{t\geqslant\tau}$, $t,\tau\in\mathbb{R}$, *has an exponential dichotomy* $\{P_t^+\}_{t\geqslant 0}$ *on* $\mathbb{R}_+$, *resp.*, $\{P_t^-\}_{t\leqslant 0}$ *on* $\mathbb{R}_-$.

Therefore, assertion (1.3) $\Rightarrow$ (i′) required for the proof of Theorems 1.1 and 1.2, follows from our next theorem (this main technical result of the current paper is proved in Section 4).

Theorem 1.6. *Assume that* X *is a reflexive Banach space, and the operator* D *is Fredholm on* $\ell_p(\mathbb{Z};X)$, $p\in[1,\infty)$, *or on* $c_0(\mathbb{Z};X)$. *Then the discrete evolution family* $\{U(n,m)\}_{n\geqslant m}$, $n,m\in\mathbb{Z}$, *has exponential dichotomies* $\{P_n^+\}_{n\geqslant 0}$ *and* $\{P_n^-\}_{n\leqslant 0}$ *on* $\mathbb{Z}_+$ *and* $\mathbb{Z}_-$, *respectively*.

Our strategy of the proof of Theorems 1.1, 1.2 and 1.6 is as follows. We will identify a family of subspaces $\{X_{n,*}^{\perp}\}_{n\in\mathbb{Z}}$ in X that is $U(n,m)$-invariant in the sense that $U(n,m)(X_{m,*}^{\perp})\subseteq X_{n,*}^{\perp}$ for $n\geqslant m$ in $\mathbb{Z}$, see Section 2. Next, we will show that the restricted evolution family $\{U(n,m)|_{X_{m,*}^{\perp}}\}_{n\geqslant m}$ has a "punctured" exponential dichotomy $\{P_n\}_{n\in\mathbb{Z}}$ on $\mathbb{Z}$, that is, we will show the following: (1) There exist projections P_n defined on $X_{n,*}^{\perp}$ that intertwine the operators $U(n,m)|_{X_{m,*}^{\perp}}$ for $n\geqslant m>0$ and for $0\geqslant n\geqslant m$; (2) the stable and unstable dichotomy estimates hold for the operators $U(n,m)|_{X_{m,*}^{\perp}}$ restricted on the subspaces $\operatorname{Im} P_m$ and $\operatorname{Ker} P_m$; and (3) there is a surjective *reduced* node operator acting from $\operatorname{Ker} P_0$ to $\operatorname{Ker} P_1$. Further, we will identify a family of subspaces in X^*, the adjoint space, such that a corresponding family of restrictions of the adjoint operators $U(n,m)^*$, $n\geqslant m$, enjoys similar properties for a family of projections $\{P_{n,*}\}_{n\in\mathbb{Z}}$ defined on certain subspaces of X^*. The punctured dichotomies just described are constructed in Section 3. To conclude the proof of Theorem 1.6, we define in Section 4 the dichotomies $\{P_n^+\}_{n\geqslant 0}$ and $\{P_n^-\}_{n\leqslant 0}$ using $\{P_n\}_{n\in\mathbb{Z}}$ and $\{(P_{n,*})^*\}_{n\in\mathbb{Z}}$. In Section 5 we finish the proof of Theorems 1.1 and 1.2. This includes a proof (based on a new approach) of the fact that (i) and (ii) in Theorem 1.1 imply (1.3), and the formulas for the defect numbers and index. Theorem 1.4 and Lemma 1.5 are proved in Section 6. Finally, in Section 7 we discuss several special cases when conditions of Theorems 1.1 and 1.2 could be easily checked, and briefly mention several classes of problems where these theorems could be applied.

2. Notation and preliminaries

Notation. We denote $\mathbb{R}_+ := \{t\in\mathbb{R} : t\geqslant 0\}$, $\mathbb{R}_- := \{t\in\mathbb{R} : t\leqslant 0\}$, $\mathbb{Z}_+ := \{n\in\mathbb{Z} : n\geqslant 0\}$, $\mathbb{Z}_- := \{n\in\mathbb{Z} : n\leqslant 0\}$, $\mathbb{T} = \{\lambda\in\mathbb{C} : |\lambda| = 1\}$; X is a Banach space; X^* is the adjoint space; A^*, $\operatorname{dom} A$, $\operatorname{Ker} A$ and $\operatorname{Im} A$ are the adjoint, domain, kernel and range of an operator A; $\sigma(A), \rho(A)$ and $\mathrm{sprad}(A)$ denote the spectrum, the resolvent set, and the spectral radius of A; the symbol $A|_Y$ denotes the restriction of A on a subspace $Y \subset X$; the Banach space of bounded linear operators from X to Y is denoted by $\mathscr{L}(X, Y)$; a generic constant is denoted by c. We use boldface to denote sequences, e.g., $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$, $x_n \in X$. For $n \in \mathbb{Z}$ the nth standard ort in $\ell_p(\mathbb{Z}; X)$ or $c_0(\mathbb{Z}; X)$ is denoted by $\mathbf{e}_n = (\delta_{nk})_{k\in\mathbb{Z}}$, where δ_{nk} is the Kronecker delta. If $x \in X$ then we denote by $x \otimes \mathbf{e}_n = (x\delta_{nk})_{k\in\mathbb{Z}}$ the sequence $x \otimes \mathbf{e}_n = (x_k)_{k\in\mathbb{Z}}$ such that $x_n = x$ and $x_k = 0$ for $k \neq n$.

For subspaces $Y \subset X$ and $Y_* \subset X^*$ we denote $Y^\perp = \{\xi \in X^* : \langle x, \xi \rangle = 0$ for all $x \in Y\}$ and $Y_*^\perp = \{x \in X : \langle x, \xi \rangle = 0$ for all $\xi \in Y_*\}$, where $\langle \cdot, \cdot \rangle$ is the (X, X^*)-pairing. If $X = X_1 \oplus X_2$, a direct sum decomposition, then we identify $(X_1)^* = X_2^\perp$ and $(X_2)^* = X_1^\perp$. If P is a projection on X, then P^* is a projection on X^* with $\mathrm{Im}\, P^* = (\mathrm{Ker}\, P)^\perp = (\mathrm{Im}\, P)^*$ and $\mathrm{Ker}\, P^* = (\mathrm{Im}\, P)^\perp = (\mathrm{Ker}\, P)^*$. If (P, Q) is a pair of projections on X, then in the direct sum decompositions $X = \mathrm{Im}\, P \oplus \mathrm{Ker}\, P$ and $X = \mathrm{Im}\, Q \oplus \mathrm{Ker}\, Q$ any operator A bounded on X can be written as the following (2×2) operator matrix:

$$A = \begin{bmatrix} P \\ I - P \end{bmatrix} A[Q \;\; I - Q] = \begin{bmatrix} PAQ & PA(I-Q) \\ (I-P)AQ & (I-P)A(I-Q) \end{bmatrix}. \tag{2.1}$$

If $AQ = PA$ then this matrix is diagonal with the diagonal entries being $A|_{\mathrm{Im}\, Q}$ and $A|_{\mathrm{Ker}\, Q}$. If $A(\mathrm{Im}\, Q) \subseteq \mathrm{Im}\, P$, or $AQ = PAQ$, then we identify $A|_{\mathrm{Im}\, Q} = AQ : \mathrm{Im}\, Q \to \mathrm{Im}\, P$, and write

$$A = \begin{bmatrix} A|_{\mathrm{Im}\, Q} & PA(I-Q) \\ 0 & (I-P)A|_{\mathrm{Ker}\, Q} \end{bmatrix}. \tag{2.2}$$

For brevity, we denote: $L_p = L_p(\mathbb{R}; X)$, $\ell_p = \ell_p(\mathbb{Z}; X)$, $\ell_{q,*} = \ell_q(\mathbb{Z}; X^*)$, $c_0 = c_0(\mathbb{Z}; X)$, $c_{0,*} = c_0(\mathbb{Z}; X^*)$, and remark that $(\ell_p)^* = \ell_{q,*}$ for $p \in [1, \infty)$, $q \in (1, \infty]$, $p^{-1} + q^{-1} = 1$, and $(c_0)^* = \ell_{1,*}$; if X is reflexive then $(c_{0,*})^* = \ell_1$.

Fibers of the kernel and cokernel of D: In Sections 2–4 we assume that the operator D from (1.5) is Fredholm on $\ell_p(\mathbb{Z}; X)$, $p \in [1, \infty)$, or on $c_0(\mathbb{Z}; X)$. Consider the operator D^* adjoint of D:

$$D^* : (\xi_n)_{n\in\mathbb{Z}} \mapsto (\xi_n - U(n+1, n)^* \xi_{n+1})_{n\in\mathbb{Z}}. \tag{2.3}$$

If the operator D is acting on ℓ_p, $p \in [1, \infty)$, resp., on c_0, then the adjoint operator D^* is acting on $\ell_{q,*}$, $q \in (1, \infty]$, resp., on $\ell_{1,*}$, and for sequences $(x_n)_{n\in\mathbb{Z}}$ and $(\xi_n)_{n\in\mathbb{Z}}$ from the spaces of X- or X^*-valued sequences we have

$$\mathrm{Ker}\, D = \{(x_n)_{n\in\mathbb{Z}} : x_n = U(n, m)x_m \text{ for all } n \geqslant m \text{ in } \mathbb{Z}\}, \tag{2.4}$$

$$\mathrm{Ker}\, D^* = \{(\xi_n)_{n\in\mathbb{Z}} : \xi_m = U(n, m)^* \xi_n \text{ for all } n \geqslant m \text{ in } \mathbb{Z}\}. \tag{2.5}$$

For each $n \in \mathbb{Z}$ we define the following subspaces:

$$X_n := \{x \in X : \text{ there exists } (x_k)_{k\in\mathbb{Z}} \in \operatorname{Ker} D \text{ so that } x = x_n\}, \tag{2.6}$$

$$X_{n,*} := \{\xi \in X^* : \text{ there exists } (\xi_k)_{k\in\mathbb{Z}} \in \operatorname{Ker} D^* \text{ so that } \xi = \xi_n\}. \tag{2.7}$$

Lemma 2.1. *For all $n \in \mathbb{Z}$ and $m \in \mathbb{Z}, m \leqslant n$, the following assertions hold*:

(i) $\dim X_n \leqslant \dim \operatorname{Ker} D < \infty$ *and* $\dim X_{n,*} \leqslant \dim \operatorname{Ker} D^* < \infty$,[2]
(ii) $U(n,m)X_m \subset X_n$; *moreover, the operator* $U(n,m)|_{X_m} : X_m \to X_n$ *is invertible*;
(iii) $U(n,m)^* X_{n,*} \subset X_{m,*}$; *moreover, the operator* $U(n,m)^*|_{X_{n,*}} : X_{n,*} \to X_{m,*}$ *is invertible*;
(iv) $U(n,m)X_{m,*}^{\perp} \subset X_{n,*}^{\perp}$ *and* $\operatorname{codim} X_{n,*}^{\perp} = \dim X_{n,*} < \infty$; $U(n,m)^* X_n^{\perp} \subset X_m^{\perp}$ *and* $\operatorname{codim} X_n^{\perp} = \dim X_n < \infty$;
(v) $X_n \subset X_{n,*}^{\perp}$ *and* $X_{n,*} \subset X_n^{\perp}$.

Proof. Assertion (i) follows from the definition of X_n and $X_{n,*}$ since D is Fredholm.

(ii) Fix $x \in X_m$, and pick a sequence $(x_k)_{k\in\mathbb{Z}} \in \operatorname{Ker} D$ such that $x = x_m$. Using (2.4), we have $x_n = U(n,m)x_m$. Since $(x_k)_{k\in\mathbb{Z}} \in \operatorname{Ker} D$, this shows that $U(n,m)x_m \in X_n$ by the definition of X_n. Since $\dim X_n < \infty$, in order to show that the operator $U(n,m)|_{X_m} : X_m \to X_n$ is invertible, it suffices to check that it is surjective. So fix an $x \in X_n$, and pick a sequence $(x_k)_{k\in\mathbb{Z}} \in \operatorname{Ker} D$ such that $x = x_n$. Using (2.4), we have $x_n = U(n,m)x_m$. By the definition of X_m, we have $x_m \in X_m$. Thus $x = U(n,m)x_m$ for some $x_m \in X_m$, and $U(n,m)|_{X_m} : X_m \to X_n$ is an isomorphism.

(iii) Exactly as in (ii), using (2.5) instead of (2.4).

(iv) For $y \in X_{m,*}^{\perp}$ we have $\langle y, \xi \rangle = 0$ for all $\xi \in X_{m,*}$. If $\eta \in X_{n,*}$ then $U(n,m)^*\eta \in X_{m,*}$ by (iii) and $\langle U(n,m)y, \eta \rangle = \langle y, U(n,m)^*\eta \rangle = 0$. Thus, $U(n,m)y \in X_{n,*}^{\perp}$. The proof for $U(n,m)^*$ is similar.

(v) Fix $x \in X_n$ and $\xi \in X_{n,*}$, and pick sequences $(x_k)_{k\in\mathbb{Z}} \in \operatorname{Ker} D$ and $(\xi_k)_{k\in\mathbb{Z}} \in \operatorname{Ker} D^*$ such that $x = x_n$ and $\xi = \xi_n$. Then

$$\begin{aligned}
\infty > \sum_{k\in\mathbb{Z}} \langle x_k, \xi_k \rangle &= \sum_{k \geqslant n} \langle x_k, \xi_k \rangle + \sum_{k<n} \langle x_k, \xi_k \rangle \\
&= \sum_{k \geqslant n} \langle U(k,n)x_n, \xi_k \rangle + \sum_{k<n} \langle x_k, U(n,k)^*\xi_n \rangle \\
&= \sum_{k \geqslant n} \langle x_n, U(k,n)^*\xi_k \rangle + \sum_{k<n} \langle U(n,k)x_k, \xi_n \rangle
\end{aligned}$$

[2] In fact, $\dim X_n = \dim \operatorname{Ker} D$ and $\dim X_{n,*} = \dim \operatorname{Ker} D^*$, see Corollary 4.1.

$$= \sum_{k \geqslant n} \langle x_n, \xi_n \rangle + \sum_{k<n} \langle x_n, \xi_n \rangle = \sum_{k \in \mathbb{Z}} \langle x, \xi \rangle,$$

where (2.4) and (2.5) have been used. Thus, $\langle x, \xi \rangle = 0$. $\square$

Invertibility of a part of D: Let $X_n' \subset X_{n,*}^{\perp}$ denote any direct complement of the finite-dimensional subspace X_n in $X_{n,*}^{\perp}$. Let Y_n denote any direct complement of the finite-codimensional subspace $X_{n,*}^{\perp}$ in X. We have the following direct sum decomposition:

$$X = X_{n,*}^{\perp} \oplus Y_n = X_n \oplus X_n' \oplus Y_n, \quad n \in \mathbb{Z}. \tag{2.8}$$

Define the following closed subspace of $\ell_p(\mathbb{Z}; X)$, $p \in [1, \infty)$, or of $c_0(\mathbb{Z}; X)$:

$$\mathscr{F} := \{(y_n)_{n \in \mathbb{Z}} : y_n \in X_{n,*}^{\perp} \text{ for each } n \in \mathbb{Z}\}. \tag{2.9}$$

Lemma 2.2. *Operator D leaves $\mathscr{F}$ invariant, and $D|_{\mathscr{F}}$ is surjective on $\mathscr{F}$.*

Proof. If $y_n \in X_{n,*}^{\perp}$ and $y_{n-1} \in X_{n-1,*}^{\perp}$ then $y_n - U(n, n-1)y_{n-1} \in X_{n,*}^{\perp}$ by Lemma 2.1(iv), and $D\mathscr{F} \subset \mathscr{F}$. To see that $D|_{\mathscr{F}}$ is surjective, we claim, first, that $\mathscr{F} \subset \operatorname{Im} D$. Since D is Fredholm, its range is closed. Therefore, $\operatorname{Im} D$ is the set of sequences $\mathbf{y}$ such that $\langle \mathbf{y}, \boldsymbol{\xi} \rangle = 0$ for all sequences $\boldsymbol{\xi} \in \operatorname{Ker} D^*$. So, to prove the claim it suffices to show that $\mathbf{y} \perp \boldsymbol{\xi}$ for all sequences $\mathbf{y} = (y_n)_{n \in \mathbb{Z}} \in \mathscr{F}$ and $\boldsymbol{\xi} = (\xi_n)_{n \in \mathbb{Z}} \in \operatorname{Ker} D^*$. If $(\xi_n)_{n \in \mathbb{Z}} \in \operatorname{Ker} D^*$ then $\xi_n \in X_{*,n}$ for all $n \in \mathbb{Z}$ by the definition of $X_{*,n}$. If $(y_n)_{n \in \mathbb{Z}} \in \mathscr{F}$ then $y_n \in X_{n,*}^{\perp}$ by the definition of $\mathscr{F}$, and the claim is proved.

Next, fix $\mathbf{y} = (y_k)_{k \in \mathbb{Z}} \in \mathscr{F} \subset \operatorname{Im} D$ and find an $\mathbf{x} = (x_k)_{k \in \mathbb{Z}} \in \ell_p(\mathbb{Z}; X)$, resp., $\mathbf{x} \in c_0(\mathbb{Z}; X)$, such that $D\mathbf{x} = \mathbf{y}$ or, in other words, such that for each $n \in \mathbb{Z}$ and all $k \in \mathbb{N}$ the following identity holds:

$$\begin{aligned} x_n &= U(n, n-1)x_{n-1} + y_n = U(n, n-1)[U(n-1, n-2)x_{n-2} + y_{n-1}] + y_n \\ &= \cdots = U(n, n-k)x_{n-k} + \sum_{j=0}^{k-1} U(n, n-j)y_j. \end{aligned}$$

To prove the surjectivity of $D|_{\mathscr{F}}$ on $\mathscr{F}$, we need to show that $x_n \in X_{n,*}^{\perp}$ for each $n \in \mathbb{Z}$. Fix $\xi \in X_{n,*}$ and pick a sequence $(\xi_k)_{k \in \mathbb{Z}} \in \operatorname{Ker} D^*$ such that $\xi = \xi_n$. By (2.5) we have $U(n, n-k)^* \xi_n = \xi_{n-k}$. Since $(y_k)_{k \in \mathbb{Z}} \in \mathscr{F}$, by Lemma 2.1(iv), we have $U(n, n-j)y_j \in X_{n,*}^{\perp}$ and $\langle U(n, n-j)y_j, \xi_n \rangle = 0$. Then

$$\begin{aligned} \langle x_n, \xi_n \rangle &= \langle x_{n-k}, U(n, n-k)^* \xi_n \rangle + \left\langle \sum_{j=0}^{k-1} U(n, n-j)y_j, \xi_n \right\rangle \\ &= \langle x_{n-k}, \xi_{n-k} \rangle \to 0 \quad \text{as } k \to \infty \end{aligned}$$

since $||x_{n-k}||\to 0$ as $k\to\infty$ for the ℓ_p-, resp., c_0-sequence $\mathbf{x}$. Thus, $\langle x_n, \xi\rangle = 0$ as claimed. $\square$

Recall that X_0' is a direct complement of X_0 in $X_{0,*}^{\perp}$, see (2.8). Define the following closed subspace $\mathscr{F}_0$ of $\mathscr{F}$, see (2.9):

$$\mathscr{F}_0 := \{(x_n)_{n\in\mathbb{Z}} \in \mathscr{F} : x_0 \in X_0'\}.$$

Let D_0 denote the restriction $D|_{\mathscr{F}}$ acting on $\mathscr{F}$ with the domain $\operatorname{dom} D_0 = \mathscr{F}_0$.

Lemma 2.3. *Operator D_0 is invertible on $\mathscr{F}$, that is, for each $(z_n)_{n\in\mathbb{Z}}\in\mathscr{F}$ there exists a unique $(x_n)_{n\in\mathbb{Z}}\in\mathscr{F}_0$ such that $D(x_n)_{n\in\mathbb{Z}} = (z_n)_{n\in\mathbb{Z}}$.*

Proof. By Lemma 2.2, for each $\mathbf{z} = (z_n)_{n\in\mathbb{Z}}\in\mathscr{F}$ there exists a sequence $\mathbf{y} = (y_n)_{n\in\mathbb{Z}}\in\mathscr{F}$ such that $D\mathbf{y} = \mathbf{z}$. By the definition of $\mathscr{F}$ we have $y_n\in X_{n,*}^{\perp}$. Using the decomposition $X_{0,*}^{\perp} = X_0\oplus X_0'$, represent $y_0 = y + y'$, where $y\in X_0$ and $y'\in X_0'$. According to the definition of X_0, there exists a sequence $(w_n)_{n\in\mathbb{Z}}\in\operatorname{Ker} D$ such that $w_0 = y$. Let $x_n = y_n - w_n$, $n\in\mathbb{Z}$. Since $y_n\in X_{n,*}^{\perp}$ and $w_n\in X_n\subset X_{n,*}^{\perp}$, see Lemma 2.1(v), we infer that $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}\in\mathscr{F}$. But $x_0 = y_0 - w_0 = y_0 - y = y'\in X_0'$. Thus $\mathbf{x}\in\mathscr{F}_0$. Since $(w_n)_{n\in\mathbb{Z}}\in\operatorname{Ker} D$, we also have $D\mathbf{x} = D\mathbf{y} = \mathbf{z}$. To prove uniqueness, assume that $\mathbf{x}\in\mathscr{F}_0$ and $\mathbf{x}\in\operatorname{Ker} D$. By the definition of X_n we have $x_n\in X_n$ for all $n\in\mathbb{Z}$. In particular, $x_0\in X_0$. But $(x_n)_{n\in\mathbb{Z}}\in\mathscr{F}_0$ means that $x_0\in X_0'$. Since $X_0\cap X_0' = \{0\}$, we have $x_0 = 0$. Since $\mathbf{x}\in\operatorname{Ker} D$, by (2.4) we conclude that $x_n = U(n,0)x_0 = 0$ for $n\geqslant 0$. Also by (2.4), we note that $0 = x_0 = U(0,n)x_n$ for $n<0$. By Lemma 2.1(ii), $U(0,n)|_{X_n} : X_n\to X_0$, $n<0$, is invertible, and thus $x_n\in X_n$ implies $x_n = 0$ for $n<0$. $\square$

3. Punctured dichotomies

Dichotomy for $U(n,m)$: We will now use the invertibility of D_0 on $\mathscr{F}$ to show that the family of the restrictions $U(n,m)|_{X_{m,*}^{\perp}} : X_{m,*}^{\perp}\to X_{n,*}^{\perp}$ has a certain exponentially dichotomic behavior on $\mathbb{Z}$ (a dichotomy on $\mathbb{Z}$ "punctured" at $m = 0$). Recall that in this section D is assumed to be Fredholm.

Proposition 3.1. *There exist a family $\{P_n\}_{n\in\mathbb{Z}}$ of projections defined on $X_{n,*}^{\perp}$ such that $\sup_{n\in\mathbb{Z}}||P_n||<\infty$, and constants $M\geqslant 1$ and $\alpha>0$ such that:*

(i) *If $n\geqslant m>0$ or if $0\geqslant n\geqslant m$, then*

$$P_nU(n,m)x = U(n,m)P_mx \quad \textit{for all } x\in X_{m,*}^{\perp}. \tag{3.1}$$

For the restriction $U(n,m)|_{\operatorname{Im} P_m}$: $\operatorname{Im} P_m\to\operatorname{Im} P_n$ we have

$$||U(n,m)|_{\operatorname{Im} P_m}||\leqslant Me^{-\alpha(n-m)}; \tag{3.2}$$

(ii) *If* $n>0\geqslant m$ *and* $x\in X_{m,*}^{\perp}$, *then* $U(n,m)P_m x = P_n U(n,0)y_0'$, *where* $y_0'\in X_0'$ *is the component of* $y=U(0,m)x$ *in the representation* $y=y_0+y_0'$, $y_0\in X_0$, *corresponding to the direct sum decomposition* $X_{0,*}^{\perp}=X_0\oplus X_0'$. *Here*, X_0' *is any direct complement of* X_0 *in* $X_{0,*}^{\perp}$;

(iii) *If* $n\geqslant m>0$ *or if* $0\geqslant n\geqslant m$ *then the restriction* $U(n,m)|_{\mathrm{Ker}\,P_m}:\ \mathrm{Ker}\,P_m\to\mathrm{Ker}\,P_n$ *is an invertible operator*, *and*

$$||(U(n,m)|_{\mathrm{Ker}\,P_m})^{-1}||\leqslant Me^{-\alpha(n-m)}.$$

(iv) *If* $n>0\geqslant m$ *then the* reduced *node operator* $N(n,m)$ *defined as*

$$N(n,m) := (I-P_n)U(n,m)|_{\mathrm{Ker}\,P_m}:\ \mathrm{Ker}\,P_m\to\mathrm{Ker}\,P_n$$

is surjective with $\mathrm{Ker}\,N(n,m)=X_m$.

Proof. Define on $\mathscr{F}$ a closed linear operator T with the domain $\mathrm{dom}\,T=\mathscr{F}_0$ by the rule $T:(x_n)_{n\in\mathbb{Z}}\mapsto(U(n,n-1)x_{n-1})_{n\in\mathbb{Z}}$, such that $D_0=I-T$. Note that although the domain of T is *not* dense in $\mathscr{F}$ (unless $X_0=\{0\}$), all standard facts from the spectral theory of closed linear operators are still valid for T (see [18, Chapter VII, Section 9]). In particular, we can use the spectrum, the resolvent set, and the resolvent of T, that is, the operator $(\lambda I-T)^{-1}$, bounded on $\mathscr{F}$, for $\lambda\in\rho(T)$.

For each $\lambda\in\mathbb{T}$, let $V(\lambda)$ denote the isometry on $\mathscr{F}$ defined by the rule $V(\lambda):(x_n)_{n\in\mathbb{Z}}\mapsto(\lambda^n x_n)_{n\in\mathbb{Z}}$. Then

$$V(\lambda^{-1})TV(\lambda)=\lambda^{-1}T,\quad |\lambda|=1. \tag{3.3}$$

Thus, $\sigma(T)=\mathbb{T}\cdot\sigma(T)$, that is, $\sigma(T)$ is rotationally invariant. Since $1\in\rho(T)$ by Lemma 2.3, we conclude that $\sigma(T)\cap\mathbb{T}=\emptyset$. Consider the Riesz projection $\mathscr{P}=(2\pi i)^{-1}\int_{|\lambda|=1}(\lambda-T)^{-1}\,d\lambda$ for T on $\mathscr{F}$ that corresponds to the part of $\sigma(T)$ inside the unit disc:

$$\sigma(T|_{\mathrm{Im}\,\mathscr{P}})=\sigma(T)\cap\{\lambda\in\mathbb{C}:|\lambda|<1\}. \tag{3.4}$$

We stress that $\mathscr{P}$ is a bounded operator on $\mathscr{F}$ and $\mathrm{Im}\,\mathscr{P}\subset\mathscr{F}_0$ since $(\lambda-T)^{-1}(x_n)_{n\in\mathbb{Z}}\in\mathrm{dom}\,T=\mathscr{F}_0$ for each $(x_n)_{n\in\mathbb{Z}}\in\mathscr{F}$ and $\lambda\in\mathbb{T}$. In addition, the operator $T\mathscr{P}$ is defined on all of $\mathscr{F}$ and is bounded, while the operator $\mathscr{P}T$ is defined only on $\mathscr{F}_0$; however, $T\mathscr{P}\supset\mathscr{P}T$, that is,

$$T\mathscr{P}(x_n)_{n\in\mathbb{Z}}=\mathscr{P}T(x_n)_{n\in\mathbb{Z}}\quad\text{for all }(x_n)_{n\in\mathbb{Z}}\in\mathscr{F}_0. \tag{3.5}$$

Also, by (3.4), $\mathrm{sprad}(T|_{\mathrm{Im}\,\mathscr{P}})<1$. The restriction $T|_{\mathrm{Ker}\,\mathscr{P}}$ is an operator on $\mathrm{Ker}\,\mathscr{P}$ with the domain $\mathrm{dom}\,T|_{\mathrm{Ker}\,\mathscr{P}}=\mathrm{Ker}\,\mathscr{P}\cap\mathscr{F}_0$ and with the spectrum $\sigma(T|_{\mathrm{Ker}\,\mathscr{P}})=$

$\sigma(T)\cap\{\lambda\in\mathbb{C}:|\lambda|>1\}$. In particular, $T|_{\operatorname{Ker}\mathscr{P}}$ is invertible in $\operatorname{Ker}\mathscr{P}$ and $\operatorname{sprad}((T|_{\operatorname{Ker}\mathscr{P}})^{-1})<1$. Fix any positive α strictly smaller than

$$-\ln\max\{\operatorname{sprad}(T|_{\operatorname{Im}\mathscr{P}}),\operatorname{sprad}((T|_{\operatorname{Ker}\mathscr{P}})^{-1})\}.$$

Thus, there is a constant $M\geqslant 1$ such that

$$\|(T|_{\operatorname{Im}\mathscr{P}})^k\|\leqslant Me^{-\alpha k}\quad\text{and}\quad\|(T|_{\operatorname{Ker}\mathscr{P}})^{-k}\|\leqslant Me^{-\alpha k},\quad k\in\mathbb{Z}_+.\tag{3.6}$$

Next, we claim that there exists a family $\{P_n\}_{n\in\mathbb{Z}}$ of projections on $X_{n,*}^{\perp}$ such that $\sup_{n\in\mathbb{Z}}\|P_n\|<\infty$ and $\mathscr{P}=\operatorname{diag}_{n\in\mathbb{Z}}[P_n]$, that is, for each $(x_n)_{n\in\mathbb{Z}}\in\mathscr{F}$ we have $\mathscr{P}(x_n)_{n\in\mathbb{Z}}=(P_nx_n)_{n\in\mathbb{Z}}$. Indeed, (3.3) and the integral formula for $\mathscr{P}$ imply $V(\lambda^{-1})\mathscr{P}V(\lambda)=\mathscr{P}$ for all $\lambda\in\mathbb{T}$. Since $\mathscr{P}$ commutes with the family $\{V(\lambda):|\lambda|=1\}$, by Baskakov [7, Lemma 3] we conclude that $\mathscr{P}$ is a diagonal operator, that is, $\mathscr{P}=\operatorname{diag}_{n\in\mathbb{Z}}[P_n]$. The operators P_n here are defined as follows: fix an $x\in X_{n,*}^{\perp}$ and define P_nx as the nth element in the sequence $\mathscr{P}(x\otimes\mathbf{e}_n)$. Note that $\sup_{n\in\mathbb{Z}}\|P_n\|=\|\mathscr{P}\|<\infty$, and the claim is proved.

Fix $m\in\mathbb{Z}$, take any $x\in X_{m,*}^{\perp}$, and let $\mathbf{x}=x\otimes\mathbf{e}_m$. Note that $\mathbf{x}\in\mathscr{F}_0$ provided either $m\neq 0$ or $m=0$ and $x\in X_0'$. If $\mathbf{x}\in\mathscr{F}_0$ then (3.5) implies

$$T\mathscr{P}\mathbf{x}=U(m+1,m)P_mx\otimes\mathbf{e}_{m+1}=\mathscr{P}T\mathbf{x}=P_{m+1}U(m+1,m)x\oplus\mathbf{e}_{m+1}.$$

Thus if $m\neq 0$, or if $m=0$ and $x\in X_0'$, then $U(m+1,m)P_mx=P_{m+1}U(m+1,m)x$. Recall that if $n>m$ then $U(n,m)=U(n,m+1)U(m+1,m)$. Using this we derive (3.1). For $\mathbf{x}=x\otimes\mathbf{e}_m$ we note that $T^j\mathbf{x}=U(m+j,m)x\otimes\mathbf{e}_{m+j}\in\mathscr{F}_0$ for $j=0,1,\dots,n-m$ provided either $n\geqslant m>0$, or $0>n\geqslant m$, or $n=0\geqslant m$ and $U(0,m)x\in X_0'$. Then the first inequality in (3.6) implies (3.2), and (i) in Proposition 3.1 is proved.

Lemma 3.2. *The following inclusions hold*:

$$X_n\subset\operatorname{Ker}P_n\ \textit{for}\ n\leqslant 0\quad\textit{and}\quad X_n\subset\operatorname{Im}P_n\ \textit{for}\ n>0.\tag{3.7}$$

Proof. We present the proof for the ℓ_p-case, the c_0-case is similar. By (2.6) and (2.4), if $x\in X_n$ then there is a sequence $(x_n)_{n\in\mathbb{Z}}\in\ell_p(\mathbb{Z};X)$ such that $x=x_n$ and $x_n=U(n,m)x_m$ for all $n\geqslant m$ in $\mathbb{Z}$. Note that $\mathscr{P}(x_n)_{n\in\mathbb{Z}}=(P_nx_n)_{n\in\mathbb{Z}}\in\operatorname{Im}\mathscr{P}\subset\mathscr{F}_0$ and thus by (3.5) we have $T^k\mathscr{P}(x_n)_{n\in\mathbb{Z}}\in\operatorname{Im}\mathscr{P}\subset\mathscr{F}_0$ for all $k\in\mathbb{N}$. If $(y_n)_{n\in\mathbb{Z}}=T^k(P_nx_n)_{n\in\mathbb{Z}}$, where $y_n=y_n(k)$, then $y_n=U(n,n-k)P_{n-k}x_{n-k}$. Using (3.1), we have that if $n-k>0$ or $0\geqslant n$ then $y_n=U(n,n-k)P_{n-k}x_{n-k}=P_nU(n,n-k)x_{n-k}$. But $(x_n)_{n\in\mathbb{Z}}\in\operatorname{Ker}D$ and thus $U(n,n-k)x_{n-k}=x_n$. So, finally,

$$y_n=P_nx_n\quad\text{for } n>k\text{ or }0\geqslant n.\tag{3.8}$$

By the first inequality in (3.6) we know that

$$\lim_{k\to\infty}\|(y_n)_{n\in\mathbb{Z}}\|_{\ell_p}=\lim_{k\to\infty}\|T^k(P_nx_n)_{n\in\mathbb{Z}}\|_{\ell_p}=0.$$

But using (3.8) we have

$$\|(y_n)_{n\in\mathbb{Z}}\|_{\ell_p}^p = \sum_{n\in\mathbb{Z}} \|y_n\|^p \geqslant \sum_{n\leqslant 0} \|y_n\|^p = \sum_{n\leqslant 0} \|P_n x_n\|^p.$$

So, $P_n x_n = 0$, that is, $X_n \subset \operatorname{Ker} P_n$ for $n \leqslant 0$.

To prove the second inclusion in (3.7), note that $((I-P_n)x_n)_{n\in\mathbb{Z}} \in \operatorname{Ker}\mathscr{P}$. Since $T|_{\operatorname{Ker}\mathscr{P}}$ is invertible on $\operatorname{Ker}\mathscr{P}$ and the second inequality in (3.6) holds, for each $k\in\mathbb{N}$ there exists a sequence $(y_n)_{n\in\mathbb{Z}} \in \mathscr{F}_0 \cap \operatorname{Ker}\mathscr{P}$, where $y_n = y_n(k)$, such that $T^k(y_n)_{n\in\mathbb{Z}} = ((I-P_n)x_n)_{n\in\mathbb{Z}}$ and

$$\lim_{k\to\infty} \|(y_n)_{n\in\mathbb{Z}}\|_{\ell_p} = \lim_{k\to\infty} \|(T|_{\operatorname{Ker}\mathscr{P}})^{-k}((I-P_n)x_n)_{n\in\mathbb{Z}}\|_{\ell_p} = 0. \tag{3.9}$$

Using the equality $x_n = U(n,m)x_m$ and (3.1), we find that if $n-k>0$ or if $0\geqslant n$ then the nth element of the sequence $T^k(y_n)_{n\in\mathbb{Z}} = ((I-P_n)x_n)_{n\in\mathbb{Z}}$ is equal to

$$\begin{aligned} U(n,n-k)y_{n-k} &= (I-P_n)x_n = (I-P_n)U(n,n-k)x_{n-k} \\ &= U(n,n-k)(I-P_{n-k})x_{n-k}. \end{aligned}$$

In other words, $y_{n-k} - (I-P_{n-k})x_{n-k} \in \operatorname{Ker} U(n,n-k)$. We claim that, in fact, this implies that

$$y_{n-k} - (I-P_{n-k})x_{n-k} = 0 \text{ provided } n>k. \tag{3.10}$$

As soon as the claim is proved, we write

$$\begin{aligned} \|(y_n)_{n\in\mathbb{Z}}\|_{\ell_p}^p &= \|(y_{n-k})_{n\in\mathbb{Z}}\|_{\ell_p}^p \geqslant \sum_{n>k} \|y_{n-k}\|^p \\ &= \sum_{n>k} \|(I-P_{n-k})x_{n-k}\|^p = \sum_{n>0} \|(I-P_n)x_n\|^p. \end{aligned}$$

Now (3.9) implies $(I-P_n)x_n = 0$, that is, $X_n \subset \operatorname{Ker} P_n$ for $n>0$. It remains to prove claim (3.10). Recall that $(y_n)_{n\in\mathbb{Z}} \in \operatorname{Ker}\mathscr{P}$ and thus $y_{n-k} - (I-P_{n-k})x_{n-k} \in \operatorname{Ker} P_{n-k}$ for $n>k$. So, it suffices to check that $\operatorname{Ker} U(n+k,n) \cap \operatorname{Ker} P_n = \{0\}$ for all $n>0$ and any $k>0$. If $n>0$ and $x \in \operatorname{Ker} U(n+k;n) \cap \operatorname{Ker} P_n$ then the sequence $\mathbf{x} = x \otimes \mathbf{e}_n$ belongs to $\operatorname{Ker}\mathscr{P} \cap \mathscr{F}_0$. Note that for $j\in\mathbb{N}$ we have $T^j\mathbf{x} = U(n+j,n)x \otimes \mathbf{e}_{n+j}$. Thus, $T^k\mathbf{x} = 0$ since $U(n+k,n)x = 0$. Now the second inequality in (3.6) implies that $0 = \|T^k\mathbf{x}\|_{\ell_p} \geqslant M^{-1}e^{\alpha k}\|\mathbf{x}\|_{\ell_p} = M^{-1}e^{\alpha k}\|x\|$. Thus, claim (3.10) is proved, and the proof of inclusions (3.7) and Lemma 3.2 is finished. $\square$

To prove (ii) in Proposition 3.1, we first consider $n=1$ and $m=0$. We can now apply (3.5) for $(x_n)_{n\in\mathbb{Z}} = x \otimes \mathbf{e}_0$ only when $x \in X_0'$, and obtain $U(1,0)P_0x = P_1U(1,0)x$ provided $x \in X_0'$. This implies that if $n>m=0$ then

$$U(n,0)P_0x = P_nU(n,0)x \quad \text{for all } x \in X_0'. \tag{3.11}$$

Next, for $n>0\geqslant m$, fix $x\in X_{m,*}^{\perp}$ and denote $y=U(0,m)x$. Using the equality $U(0,m)P_mx=P_0U(0,m)x$ from (3.1), we have $U(n,m)P_mx=U(n,0)U(0,m)P_mx=U(n,0)P_0y$. Represent $y=y_0+y_0'$, where $y_0\in X_0$ and $y_0'\in X_0'$, and recall that $P_0y_0=0$ by (3.7) in Lemma 3.2. Then, using Eq. (3.11), we conclude: $U(n,m)P_mx=U(n,0)P_0(y_0+y_0')=U(n,0)P_0y_0'=P_nU(n,0)y_0'$, and (ii) in Proposition 3.1 is proved.

To prove (iii) in Proposition 3.1, remark that by the second inequality in (3.6) we have the inequality $\|(T|_{\mathrm{Ker}\,\mathscr{P}})^{-k}(x_n)_{n\in\mathbb{Z}}\|_{\mathscr{F}}\leqslant Me^{-\alpha k}\|(x_n)_{n\in\mathbb{Z}}\|_{\mathscr{F}}$. As soon as $T^j(x_n)_{n\in\mathbb{Z}}\in\mathrm{Ker}\,\mathscr{P}\cap\mathscr{F}_0$ for $j=0,1,\dots,k-1$, we then have $\|T^k(x_n)_{n\in\mathbb{Z}}\|_{\mathscr{F}}\geqslant M^{-1}e^{\alpha k}\|(x_n)_{n\in\mathbb{Z}}\|_{\mathscr{F}}$. In particular, $T^j(x\otimes\mathbf{e}_m)=U(n+j,m)x\otimes\mathbf{e}_{m+j}\in\mathscr{F}_0$ if and only if either $m>0$, or $m+j<0$, or $m=-j$ and $U(0,-j)x\in X_0'$. This implies that $\|U(m+k,m)x\|\geqslant M^{-1}e^{\alpha k}\|x\|$ provided one of the following three possibilities hold: (a) $m>0$, $k\in\mathbb{Z}_+$, $x\in\mathrm{Ker}\,P_m$; (b) $m<0$, $k=0,1,\dots,-m$, $x\in\mathrm{Ker}\,P_m$; (c) $m=0$, $k\in\mathbb{Z}_+$, $x\in X_0'\cap\mathrm{Ker}\,P_0$. This proves (iii).

To prove (iv) in Proposition 3.1, we first consider the reduced node operator $N(1,0)=(I-P_1)U(1,0)|_{\mathrm{Ker}\,P_0}:\mathrm{Ker}\,P_0\to\mathrm{Ker}\,P_1$. Note that $\mathrm{Ker}\,N(1,0)=\{x\in\mathrm{Ker}\,P_0:U(1,0)x\in\mathrm{Im}\,P_1\}$. We claim that $X_0=\mathrm{Ker}\,N(1,0)$. Indeed, $U(1,0)(X_0)=X_1\subset\mathrm{Im}\,P_1$ by Lemma 2.1(ii) and (3.7), which implies $X_0\subset\mathrm{Ker}\,N(1,0)$. To prove the inverse inclusion, assume that $x\in\mathrm{Ker}\,P_0$ and $U(1,0)x\in\mathrm{Im}\,P_1$. Using $X_{0,*}^{\perp}=X_0\oplus X_0'$, decompose $x=x_0+x_0'$. Then $U(1,0)x_0'=U(1,0)x-U(1,0)x_0\in\mathrm{Im}\,P_1$ since $U(1,0)x\in\mathrm{Im}\,P_1$ by assumption and $U(1,0)x_0\in X_1\subset\mathrm{Im}\,P_1$ by Lemma 2.1(ii) and (3.7). Also, $x_0'\in\mathrm{Ker}\,P_0\cap X_0'$ since $x_0'=x-x_0$ and $x\in\mathrm{Ker}\,P_0$ by assumption, and $x_0\in X_0\subset\mathrm{Ker}\,P_0$ by (3.7). Therefore, $x_0'\otimes\mathbf{e}_0\in\mathrm{Ker}\,\mathscr{P}\cap\mathscr{F}_0$ and, using (3.6), we obtain for $k\in\mathbb{N}$:

$$\begin{aligned}\|U(k,1)U(1,0)x_0'\|&=\|U(k,0)x_0'\|=\|U(k,0)x_0'\otimes\mathbf{e}_k\|_{\ell_p}\\&=\|T^k(x_0'\otimes\mathbf{e}_0)\|_{\ell_p}\geqslant M^{-1}e^{\alpha k}\|x_0'\otimes\mathbf{e}_0\|_{\ell_p}=M^{-1}e^{\alpha k}\|x_0'\|.\end{aligned}$$

But then (3.2) for $U(1,0)x_0'\in\mathrm{Im}\,P_1$ implies $\|x_0'\|=0$ and thus $x=x_0$ proving $\mathrm{Ker}\,N(1,0)\subset X_0$.

Next, we show that for each $y\in\mathrm{Ker}\,P_1$ there is an $x\in\mathrm{Ker}\,P_0$ such that $(I-P_1)U(1,0)x=y$. Take $y\otimes\mathbf{e}_1\in\mathrm{Ker}\,\mathscr{P}$ and find $(x_n)_{n\in\mathbb{Z}}\in\mathrm{Ker}\,\mathscr{P}\cap\mathscr{F}_0$ so that $T(x_n)_{n\in\mathbb{Z}}=y\otimes\mathbf{e}_1$. In particular, $U(1,0)x_0=y$ for $x_0\in\mathrm{Ker}\,P_0\cap X_0'$. Then $y=(I-P_1)y=(I-P_1)U(1,0)x_0$, and $N(1,0)$ is surjective from $\mathrm{Ker}\,P_0$ to $\mathrm{Ker}\,P_1$ with $\mathrm{Ker}\,N(1,0)=X_0$.

To finish the proof of (iv) in Proposition 3.1 for any $n>0\geqslant m$, we remark that $U(n,m)=U(n,1)U(1,0)U(0,m)$ and (3.1) imply: $(I-P_n)U(n,m)(I-P_m)=[(I-P_n)U(n,1)(I-P_1)]N(1,0)[(I-P_0)U(0,m)(I-P_m)]$. Operators in brackets are invertible by (iii), and the general case $n>0\geqslant m$ in (iv) follows from the case $n=1$ and $m=0$ proved above. $\square$

Dichotomy for $U(n,m)^*$: In addition to Proposition 3.1, for the proof of Theorem 1.6 we will need to consider the following dual objects. For $k\geqslant\ell$ in $\mathbb{Z}$ define an exponentially bounded evolution family $\{U_*(k,\ell)\}_{k\geqslant\ell}$ on X^* by $U_*(k,\ell) = U(-\ell,-k)^*$. Let $D_* : (\xi_k)_{k\in\mathbb{Z}} \mapsto (\xi_k - U_*(k,k-1)\xi_{k-1})_{k\in\mathbb{Z}}$ denote the corresponding difference operator. Also, consider an operator, $D_\sharp$, defined by the rule $D_\sharp : (\xi_n)_{n\in\mathbb{Z}} \mapsto (\xi_n - U(n+1,n)^*\xi_{n+1})_{n\in\mathbb{Z}}$ on the following spaces: If D is acting on ℓ_p, $p\in(1,\infty)$, then $D_\sharp$ is considered on $\ell_{q,*}$, $q\in(1,\infty)$, $p^{-1}+q^{-1}=1$, and then $D_\sharp = D^*$, the adjoint operator of D. If D is acting on ℓ_1, then $D_\sharp$ is considered on $c_{0,*}$, and then $(D_\sharp)^* = D$. If D is acting on c_0, then $D_\sharp$ is considered on $\ell_{1,*}$, and then $D_\sharp = D^*$. If $j : (\xi_k)_{k\in\mathbb{Z}} \mapsto (\xi_{-k})_{k\in\mathbb{Z}}$, and the operator D_* is considered on the same sequence space as $D_\sharp$, then $D_* = jD_\sharp j^{-1}$. Since D is Fredholm if and only if D^* is Fredholm, we infer that $D_\sharp$ is Fredholm, and therefore D_* is Fredholm. Moreover, $\operatorname{ind} D_* = \operatorname{ind} D$. Apply (2.4) and (2.5) for D_* and $\{U_*(k,\ell)\}_{k\geqslant\ell}$, and remark that $U_*(k,\ell)^* = U(-\ell,-k)$ acts on X by the reflexivity assumption. Then, for sequences $(\xi_k)_{k\in\mathbb{Z}}$ and $(z_k)_{k\in\mathbb{Z}}$ from the corresponding sequence spaces, we infer

$$\operatorname{Ker} D_* = \{(\xi_k)_{k\in\mathbb{Z}} : \xi_k = U_*(k,\ell)\xi_\ell,\ k\geqslant\ell\}, \tag{3.12}$$

$$\operatorname{Ker}(D_*)^* = \{(z_k)_{k\in\mathbb{Z}} : z_\ell = U_*(k,\ell)^* z_k,\ k\geqslant\ell\}. \tag{3.13}$$

For $k\in\mathbb{Z}$ introduce subspaces $Z_{k,*}\subset X^*$, resp. $Z_k\subset X$, resp. $Z_k^\perp\subset X^*$, for $\{U_*(k,\ell)\}_{k\geqslant\ell}$ that are analogous to the subspaces $X_n\subset X$, resp. $X_{n,*}\subset X^*$, resp. $X_{n,*}^\perp\subset X$, for $\{U(n,m)\}_{n\geqslant m}$, defined in (2.6) and (2.7):

$$Z_{k,*} = \{\xi\in X^* :\ \text{there exists } (\xi_\ell)_{\ell\in\mathbb{Z}}\in\operatorname{Ker} D_* \text{ so that } \xi = \xi_k\}, \tag{3.14}$$

$$Z_k = \{z\in X :\ \text{there exists } (z_\ell)_{\ell\in\mathbb{Z}}\in\operatorname{Ker}(D_*)^* \text{ so that } z = z_k\}. \tag{3.15}$$

Lemma 3.3. *For each $k\in\mathbb{Z}$ we have $Z_k = X_{-k}$ and $Z_{k,*} = X_{-k,*}$.*

Proof. By formulas (3.13) and (3.15), $z\in Z_k$ if and only if $z = z_k$ for a sequence $(z_\ell)_{\ell\in\mathbb{Z}}$ such that $z_\ell = U_*(k,\ell)^* z_k = (U(-\ell-k)^*)^* z_k = U(-\ell,-k)z_k$ for all $k\geqslant\ell$. By formulas (2.4) and (2.6), $x\in X_m$ if and only if $x = x_m$ for a sequence $(x_n)_{n\in\mathbb{Z}}$ such that $x_n = U(n,m)x_m$ for all $n\geqslant m$. Setting $z_{-n} = x_n$, $n\in\mathbb{Z}$, thus proves $Z_k = X_{-k}$. The proof of $Z_{k,*} = X_{-k,*}$ is similar. $\square$

Apply Proposition 3.1 to the evolution family $\{U_*(k,\ell)\}_{k\geqslant\ell}$. This proposition gives the following assertions: a dichotomy for the restriction $\{U_*(k,\ell)|_{Z_\ell^\perp}\}_{k\geqslant\ell}$ for $k\geqslant\ell>0$ and $0\geqslant k\geqslant\ell$, an analogue of Lemma 3.2, and the surjectivity of the reduced node operator that corresponds to this restriction. Using Lemma 3.3, and setting $n=-\ell$ and $m=-k$ for $n\geqslant m$ in $\mathbb{Z}$, we now recast these assertions for the family $\{U(n,m)^*|_{X_n^\perp}\}_{n\geqslant m}$ as follows (cf. Proposition 3.1 and Lemma 3.2).

Proposition 3.4. *There exist a family $\{P_{n,*}\}_{n\in\mathbb{Z}}$ of projections defined on $X_n^{\perp}$ such that $\sup_{n\in\mathbb{Z}} \|P_{n,*}\| < \infty$, and constants $M \geqslant 1$ and $\alpha > 0$ such that*:

(i) *If $n \geqslant m \geqslant 0$ or if $0 > n \geqslant m$ then*

$$P_{m,*}U(n,m)^*\xi = U(n,m)^*P_{n,*}\xi \quad \textit{for all } \xi \in X_n^{\perp}. \tag{3.16}$$

For the restriction $U(n,m)^*|_{\mathrm{Im}\,P_{n,*}} : \mathrm{Im}\,P_{n,*} \to \mathrm{Im}\,P_{m,*}$ we have

$$\|U(n,m)^*|_{\mathrm{Im}\,P_{n,*}}\| \leqslant Me^{-\alpha(n-m)}. \tag{3.17}$$

(ii) *If $n \geqslant 0 > m$ and $\xi \in X_n^{\perp}$, then*

$$U(n,m)^*P_{n,*}\xi = P_{m,*}U(0,m)^*\zeta_0', \tag{3.18}$$

where $\zeta_0' \in X_{0,*}'$ is the component of $\zeta = U(n,0)^*\xi$ in the representation $\zeta = \zeta_0 + \zeta_0'$, $\zeta_0 \in X_{0,*}$, corresponding to the direct sum decomposition $X_0^{\perp} = X_{0,*} \oplus X_{0,*}'$. Here, $X_{0,*}'$ is any direct complement of $X_{0,*}$ in $X_0^{\perp}$.

(iii) *If $n \geqslant m \geqslant 0$ or if $0 > n \geqslant m$ then the restriction $U(n,m)^*|_{\mathrm{Ker}\,P_{n,*}} : \mathrm{Ker}\,P_{n,*} \to \mathrm{Ker}\,P_{m,*}$ is an invertible operator, and*

$$\|(U(n,m)^*|_{\mathrm{Ker}\,P_{n,*}})^{-1}\| \leqslant Me^{-\alpha(n-m)}. \tag{3.19}$$

(iv) *If $n \geqslant 0 > m$ then the reduced node operator $N_*(n,m)$ defined as*

$$N_*(n,m) = (I - P_{m,*})U(n,m)^*|_{\mathrm{Ker}\,P_{n,*}} : \mathrm{Ker}\,P_{n,*} \to \mathrm{Ker}\,P_{m,*} \tag{3.20}$$

is surjective with $\mathrm{Ker}\,N_*(n,m) = X_{n,*}$.

(v) *The following inclusions hold*:

$$X_{n,*} \subset \mathrm{Ker}\,P_{n,*} \textit{ for } n \geqslant 0 \quad \textit{and} \quad X_{n,*} \subset \mathrm{Im}\,P_{n,*} \textit{ for } n < 0. \tag{3.21}$$

Invariant direct complements: Recall the direct sum decomposition $X = X_{n,*}^{\perp} \oplus Y_n$, see (2.8). It allows us to identify

$$(Y_n)^* = (X_{n,*}^{\perp})^{\perp} = X_{n,*}, \quad n \in \mathbb{Z}. \tag{3.22}$$

Recall that $\dim X_{n,*} < \infty$ by Lemma 2.1(i) and thus $X_{n,*}$ has a direct complement in X^*. Let $Q_{n,*}$ be a bounded projection on X^* such that $\mathrm{Im}\,Q_{n,*} = X_{n,*}$. By Lemma 2.1(iii) we have $U(n,m)^*(X_{n,*}) \subseteq X_{m,*}$, $n \geqslant m$, or

$$U(n,m)^*Q_{n,*} = Q_{m,*}U(n,m)^*Q_{n,*}. \tag{3.23}$$

Note that Y_n is an arbitrary direct complement of the finitely codimensional subspace $X_{n,*}^{\perp}$ in X, and, generally, $U(n,m)(Y_m) \nsubseteq Y_n$. Using representation (2.2) with $P = Q_{m,*}$ and $Q = Q_{n,*}$ for $A = U(n,m)^*$ in the decompositions $X^* = \operatorname{Im} Q_{m,*} \oplus \operatorname{Ker} Q_{m,*}$ and $X^* = \operatorname{Im} Q_{n,*} \oplus \operatorname{Ker} Q_{n,*}$, we will identify the restriction $U(n,m)^*|_{X_{n,*}}$ and the operator $U(n,m)^* Q_{n,*} : X_{n,*} \to X_{m,*}$. This is a finite dimensional and, by Lemma 2.1(iii), invertible operator. By (3.22) and (3.23), $(U(n,m)^* Q_{n,*})^* = Q^*_{n,*} U(n,m) Q^*_{m,*} : Y_m \to Y_n$.

If $n \geqslant 0$ then $X_{n,*} \subset \operatorname{Ker} P_{n,*}$ by (3.21) and thus (3.19) implies

$$||U(n,m)^* \xi|| \geqslant M^{-1} e^{\alpha(n-m)} ||\xi|| \quad \text{for all } \xi \in X_{n,*}. \tag{3.24}$$

Hence, $||(U(n,m)^* Q_{n,*})^{-1}||_{\mathscr{L}(X_{m,*}, X_{n,*})} \leqslant M e^{-\alpha(n-m)}$. Passing to the adjoint in (3.23), and using (3.22), we conclude that the operator

$$Q^*_{n,*} U(n,m) = Q^*_{n,*} U(n,m) Q^*_{m,*} : Y_m \to Y_n \tag{3.25}$$

is invertible, and

$$||(Q^*_{n,*} U(n,m) Q^*_{m,*})^{-1}||_{\mathscr{L}(Y_n, Y_m)} \leqslant M e^{-\alpha(n-m)}, \quad n \geqslant m \geqslant 0, \tag{3.26}$$

for any direct complement Y_n of $X^{\perp}_{n,*}$ in X. Next, we will identify the direct complement of $X^{\perp}_{n,*}$ in $X, n \geqslant 0$, which is $U(n,m)$-invariant. Fix any Y_0 such that $X^{\perp}_{0,*} \oplus Y_0 = X$. For each $n \geqslant 0$ define $W_n := \{U(n,0) y_0 : y_0 \in Y_0\}$.

Lemma 3.5. *For all $n \geqslant m \geqslant 0$ in $\mathbb{Z}_+$ the following assertions hold*:

(i) *the subspace W_n is closed*;
(ii) $X^{\perp}_{n,*} \oplus W_n = X$;
(iii) $U(n,m) W_m \subseteq W_n, \; n \geqslant m \geqslant 0$;
(iv) *the restriction $U(n,m)|_{W_m} : W_m \to W_n$ is invertible, and*

$$||(U(n,m)|_{W_m})^{-1}|| \leqslant M e^{-\alpha(n-m)}. \tag{3.27}$$

Proof. (i) Inequalities (3.26) and (3.25) for $n \geqslant m = 0$ imply for all $y_0 \in Y_0$:

$$M^{-1} e^{\alpha n} ||y_0|| \leqslant ||Q^*_{n,*} U(n,0) Q^*_{0,*} y_0|| = ||Q^*_{n,*} U(n,0) y_0|| \leqslant ||Q^*_{n,*}|| \, ||U(n,0) y_0||. \tag{3.28}$$

Thus, $||U(n,0) y_0|| \geqslant c ||y_0||$ for some $c > 0$, and (i) holds.

(ii) To see $X^{\perp}_{n,*} \cap W_n = \{0\}$, assume that $x = U(n,0) y_0 \in X^{\perp}_{n,*}$ for some $y_0 \in Y_0$. Since $U(n,0)^* : X_{n,*} \to X_{0,*}$ is an isomorphism by Lemma 2.1(iii), if $\xi_0 \in X_{0,*}$ then $\xi_0 = U(n,0)^* \xi_n$ for some $\xi_n \in X_{n,*}$. Since $x \in X^{\perp}_{n,*}$, for each $\xi_0 \in X_{0,*}$ we have: $\langle y_0, \xi_0 \rangle = \langle y_0, U(n,0)^* \xi_n \rangle = \langle U(n,0) y_0, \xi_n \rangle = \langle x, \xi_n \rangle = 0$. Thus, $y_0 \in X^{\perp}_{0,*} \cap Y_0$ and $y_0 = 0 = x$.

To see $(W_n + X^{\perp}_{n,*})^{\perp} = W^{\perp}_n \cap X_{n,*} = \{0\}$, assume that $\xi_n \in W^{\perp}_n \cap X_{n,*}$. Then for each $y_0 \in Y_0$ and $x = U(n,0) y_0 \in W_n$ we have $0 = \langle \xi_n, x \rangle = \langle \xi_n, U(n,0) y_0 \rangle = \langle U(n,0)^* \xi_n, y_0 \rangle$. Thus, $U(n,0)^* \xi_n \in (Y_0)^{\perp}$. On the other hand, $\xi_n \in X_{n,*}$ and

Lemma 2.1(iii) imply $U(n,0)^*\xi_n \in X_{0,*}$. Thus $U(n,0)^*\xi_n = 0$ and $\xi_n = 0$ by Lemma 2.1(iii), which finishes the proof of (ii).

(iii) If $x = U(m,0)y_0 \in W_m$ then $U(n,m)x = U(n,0)y_0 \in W_n$.

(iv) By (ii), we have $(W_n)^* = (X_{n,*}^{\perp})^{\perp} = X_{n,*}$. By (iii), we are in the situation when $U(n,m)^*|_{X_{n,*}} : X_{n,*} \to X_{m,*}$ is the adjoint of the operator $U(n,m)|_{W_m} : W_m \to W_n$. By (3.24), both (finite-dimensional) operators are invertible, the norms of inverses are equal, and thus (3.24) implies (3.27). $\square$

We proceed further with a construction of the direct complement of $X_n^{\perp}$, $n \leqslant 0$, in X^* which is $U(n,m)^*$-invariant. Consider a direct sum decomposition $X^* = X_n^{\perp} \oplus Y_{n,*}$, $n \leqslant 0$, where $Y_{n,*}$ is any direct complement of the (finitely codimensional) subspace $X_n^{\perp}$ in X^*. We may identify $(Y_{n,*})^* = (X_n^{\perp})^{\perp} = X_n$. Define $W_{n,*} = \{U(0,n)^*\xi_0 : \xi_0 \in Y_{0,*}\}$, $n \leqslant 0$.

Lemma 3.6. *For all $m \leqslant n \leqslant 0$ in $\mathbb{Z}_-$ the following assertions hold*:

(i) *the subspace $W_{n,*}$ is closed*;
(ii) $X_n^{\perp} \oplus W_{n,*} = X^*$;
(iii) $U(n,m)^* W_{n,*} \subseteq W_{m,*}$;
(iv) *the restriction $U(n,m)^*|_{W_{n,*}} : W_{n,*} \to W_{m,*}$ is invertible, and*

$$||(U(n,m)^*|_{W_{n,*}})^{-1}|| \leqslant Me^{-\alpha(n-m)}. \tag{3.29}$$

Proof. The proof is parallel to the proof of Lemma 3.5. Indeed, the inclusion $X_n \subset \operatorname{Ker} P_n$, $n \leqslant 0$, in (3.7) and Proposition 3.1(iii) imply that $U(0,n)|_{X_n} : X_n \to X_0$ is invertible with $||(U(0,n)|_{X_n})^{-1}|| \leqslant Me^{\alpha n}$, $n \leqslant 0$. Using any bounded projection Q_n on X with $\operatorname{Im} Q_n = X_n$, we identify $U(0,n)|_{X_n} = U(0,n)Q_n = Q_0 U(0,n)Q_n$. Passing to the adjoint operator, cf. (3.28), we conclude that $||U(0,n)^*\xi|| \geqslant c||\xi||$ for all $\xi \in Y_{0,*} = (X_0)^*$. This gives (i), and the proof of (ii)–(iv) is identical (dual) to the proof of Lemma 3.5. $\square$

4. Proof of Theorem 1.6

Proof of Theorem 1.6. For $n \geqslant 0$: First, consider $n > 0$. By Proposition 3.1 and Lemma 3.5(ii) we have a direct sum decomposition $X = X_{n,*}^{\perp} \oplus W_n = \operatorname{Im} P_n \oplus \operatorname{Ker} P_n \oplus W_n$, $n > 0$. Let P_n^+ be a projection on X with

$$\operatorname{Im} P_n^+ = \operatorname{Im} P_n \quad \text{and} \quad \operatorname{Ker} P_n^+ = \operatorname{Ker} P_n \oplus W_n, \quad n > 0. \tag{4.1}$$

For $n \geqslant m > 0$, if $x \in \operatorname{Im} P_m^+$ then $U(n,m)x \in \operatorname{Im} P_n^+$ by (3.1). If $x = y + z \in \operatorname{Ker} P_m^+$, where $y \in \operatorname{Ker} P_m$, $z \in W_m$, then $U(n,m)x = U(n,m)y + U(n,m)z \in \operatorname{Ker} P_n^+$ by (3.1) and Lemma 3.5(iii). This gives $U(n,m)P_m^+ = P_n^+ U(n,m)$ for $n \geqslant m > 0$. From (3.2) we infer

$$\|U(n,m)|_{\mathrm{Im}\, P_m^+}\| = \|U(n,m)|_{\mathrm{Im}\, P_m}\| \leqslant Me^{-\alpha(n-m)}, \quad n \geqslant m > 0.$$

The matrix representation (2.2) of the operator $A = U(n,m)|_{\mathrm{Ker}\, P_m^+}$ in the decompositions $\mathrm{Ker}\, P_m^+ = \mathrm{Ker}\, P_m \oplus W_m$ and $\mathrm{Ker}\, P_n^+ = \mathrm{Ker}\, P_n \oplus W_n$ is diagonal by (3.1) and Lemma 3.5(iii) with the invertible diagonal blocks $U(n,m)|_{\mathrm{Ker}\, P_m}$ and $U(n,m)|_{W_m}$. Then the operator $U(n,m)|_{\mathrm{Ker}\, P_m}$ is invertible; its inverse satisfies the estimate in Proposition 3.1(iii). The operator $U(n,m)|_{W_m}$ satisfies (3.27). Thus, we have $\|(U(n,m)|_{\mathrm{Ker}\, P_m^+})^{-1}\| \leqslant Me^{-\alpha(n-m)}$ for $n \geqslant m > 0$.

Next, consider $n = 0$. Recall that X_0' is a direct complement of X_0 in $X_{0,*}^{\perp}$, and that $X_0 \subset \mathrm{Ker}\, P_0$ by (3.7) and $\mathrm{Ker}\, P_0 \subset X_{0,*}^{\perp}$ by Proposition 3.1. Denote $\tilde{X}_0 = X_0' \cap \mathrm{Ker}\, P_0$. For each $x \in \mathrm{Ker}\, P_0$ use the direct sum decomposition $X_{0,*}^{\perp} = X_0 \oplus X_0'$ to write $x = x_0 + x_0'$ with unique $x_0 \in X_0$, $x_0' \in X_0'$. Then $x_0' = x - x_0 \in \mathrm{Ker}\, P_0$ and thus $x_0' \in \tilde{X}_0$. So, $\tilde{X}_0$ is a direct complement of X_0 in $\mathrm{Ker}\, P_0$, that is, $X_0 \oplus \tilde{X}_0 = \mathrm{Ker}\, P_0$. We claim that

$$U(1,0) : \tilde{X}_0 \to \mathrm{Ker}\, P_1 \text{ is an isomorphism.} \tag{4.2}$$

Indeed, if $x \in \mathrm{Ker}\, P_1$ then, by the surjectivity of the node operator $N(1,0)$ from Proposition 3.1(iv) there exists $y \in \mathrm{Ker}\, P_0$ so that $N(1,0)y = (I - P_1)U(1,0)y = x$. Use the direct sum decomposition $\mathrm{Ker}\, P_0 = X_0 \oplus \tilde{X}_0$ to write $y = y_0 + \tilde{y}_0$, where $y_0 \in X_0$, $\tilde{y}_0 \in \tilde{X}_0$. Since $\mathrm{Ker}\, N(1,0) = X_0$, we have $x = N(1,0)y = N(1,0)\tilde{y}_0 = (I - P_1)U(1,0)\tilde{y}_0$. Since $\tilde{y}_0 \in \tilde{X}_0 \subset \mathrm{Ker}\, P_0$, we have $P_0\tilde{y}_0 = 0$. But $\tilde{y}_0 \in \tilde{X}_0 \subset X_0'$, and (3.11) then implies $0 = U(1,0)P_0\tilde{y}_0 = P_1 U(1,0)\tilde{y}_0$. Thus, $U(1,0)\tilde{y}_0 \in \mathrm{Ker}\, P_1$, and $U(1,0)\tilde{y}_0 = (I - P_1)U(1,0)\tilde{y}_0 = x$. Therefore, $U(1,0) : \tilde{X}_0 \to \mathrm{Ker}\, P_1$ is surjective. Next, if $U(1,0)\tilde{y}_0 = 0$ for some $\tilde{y}_0 \in \tilde{X}_0 \subset \mathrm{Ker}\, P_0$, then $N(1,0)\tilde{y}_0 = 0$. Since $\mathrm{Ker}\, N(1,0) = X_0$ by Proposition 3.1(iv), we have $\tilde{y}_0 \in X_0$ and thus $\tilde{y}_0 = 0$ since $X_0 \cap \tilde{X}_0 = \{0\}$. This proves (4.2).

Define a projection P_0^+ on X such that

$$\mathrm{Im}\, P_0^+ = \mathrm{Im}\, P_0 \oplus X_0 \quad \text{and} \quad \mathrm{Ker}\, P_0^+ = Y_0 \oplus \tilde{X}_0 \tag{4.3}$$

so that $X = \mathrm{Im}\, P_0^+ \oplus \mathrm{Ker}\, P_0^+$ by (2.8) and $X_{0,*}^{\perp} = \mathrm{Ker}\, P_0 \oplus \mathrm{Im}\, P_0$ by Proposition 3.1. Recall that $\mathrm{Im}\, P_1^+ = \mathrm{Im}\, P_1$ and $\mathrm{Ker}\, P_1^+ = W_1 \oplus \mathrm{Ker}\, P_1$, see (4.1). Note that we have $U(1,0)(X_0) \subseteq X_1 \subset \mathrm{Im}\, P_1$ by Lemma 2.1(ii) and (3.7). Also,

$$U(1,0)(\mathrm{Im}\, P_0) \subset \mathrm{Im}\, P_1. \tag{4.4}$$

Indeed, using Proposition 3(ii), we have that if $x = P_0 x$ then $U(1,0)x = U(1,0)P_0 x = P_1 U(1,0)y_0' \in \mathrm{Im}\, P_1$. Thus, $U(1,0)\mathrm{Im}\, P_0^+ \subset \mathrm{Im}\, P_1^+$. Also, we have that $U(1,0)(Y_0) = W_1 \subset \mathrm{Ker}\, P_1^+$ by Lemma 3.5(iii) and $U(1,0)(\tilde{X}_0) = \mathrm{Ker}\, P_1 \subset \mathrm{Ker}\, P_1^+$ by claim (4.2). This proves $U(1,0)(\mathrm{Ker}\, P_0^+) \subset \mathrm{Ker}\, P_1^+$ and $U(1,0)P_0^+ = P_1^+ U(1,0)$.

For $n \geqslant 2$ and $x \in \operatorname{Im} P_0^+$ we have $\|U(n,0)x\| = \|U(n,1)U(1,0)x\| \leqslant Me^{-\alpha(n-1)}\|U(1,0)x\| \leqslant M'e^{-\alpha n}\|x\|$ because $U(1,0)x \in \operatorname{Im} P_1^+$. Also, the restriction $U(n,0)|_{\operatorname{Ker} P_0^+} = U(n,1)|_{\operatorname{Ker} P_1^+} U(1,0)|_{\operatorname{Ker} P_0^+}$ is invertible from $\operatorname{Ker} P_0^+$ to $\operatorname{Ker} P_n^+$. Indeed, $U(n,1)|_{\operatorname{Ker} P_1^+} : \operatorname{Ker} P_1^+ \to \operatorname{Ker} P_n^+$ is invertible by the proof of dichotomy for $n \geqslant 1$. Also, $U(1,0)|_{\operatorname{Ker} P_0^+} : \operatorname{Ker} P_0^+ \to \operatorname{Ker} P_1^+$ is a direct sum of two operators, $U(1,0)|_{Y_0} : Y_0 \to W_1$ and $U(1,0)|_{\tilde{X}_0} : \tilde{X}_0 \to \operatorname{Ker} P_1$. The first operator is invertible by Lemma 3.5(iv) and the second operator is invertible by claim (4.2). Exponential estimates for $\|(U(n,0)|_{\operatorname{Ker} P_0^+})^{-1}\|$ follow from the estimates for $\|(U(n,1)|_{\operatorname{Ker} P_1^+})^{-1}\|$. $\square$

Proof of Theorem 1.6. For $n \leqslant 0$: It is convenient to work on X^* with the family $\{U(n,m)^*\}_{0 \geqslant n \geqslant m}$. First, consider $n<0$. By Proposition 3.4 and Lemma 3.6(ii) we have the direct sum decomposition $X^* = X_n^\perp \oplus W_{n,*} = \operatorname{Im} P_{n,*} \oplus \operatorname{Ker} P_{n,*} \oplus W_{n,*}$, $n<0$. Let $R_{n,*}$ be a projection on X^* such that

$$\operatorname{Im} R_{n,*} = \operatorname{Im} P_{n,*} \quad \text{and} \quad \operatorname{Ker} R_{n,*} = \operatorname{Ker} P_{n,*} \oplus W_{n,*}, \quad n<0. \tag{4.5}$$

As in the proof of Theorem 1.6 for $n>0$, one checks for $0>n \geqslant m$ the following assertions:

$$U(n,m)^* R_{n,*} = R_{m,*} U(n,m)^*, \tag{4.6}$$

$$\|U(n,m)^*|_{\operatorname{Im} R_{n,*}}\| \leqslant Me^{-\alpha(n-m)}, \tag{4.7}$$

the restriction $U(n,m)^*|_{\operatorname{Ker} R_{n,*}} : \operatorname{Ker} R_{n,*} \to \operatorname{Ker} R_{m,*}$ is invertible, and

$$\|(U(n,m)^*|_{\operatorname{Ker} R_{n,*}})^{-1}\| \leqslant Me^{-\alpha(n-m)}. \tag{4.8}$$

Next, consider $n=0$. Let $X'_{0,*}$ be a direct complement of $X_{0,*}$ in $X_0^\perp$ and recall that $X_{0,*} \subset \operatorname{Ker} P_{0,*}$ by (3.21). Denote $\tilde{X}_{0,*} = X'_{0,*} \cap \operatorname{Ker} P_{0,*}$, so that $\operatorname{Ker} P_{0,*} = X_{0,*} \oplus \tilde{X}_{0,*}$. Define a projection $R_{0,*}$ on X^* as follows:

$$\operatorname{Im} R_{0,*} = \operatorname{Im} P_{0,*} \oplus X_{0,*}, \quad \operatorname{Ker} R_{0,*} = Y_{0,*} \oplus \tilde{X}_{0,*}. \tag{4.9}$$

We now prove that assertions (4.6)–(4.8) hold for $0 \geqslant n \geqslant m$ (cf. the corresponding part of the proof of Theorem 1.6 for $n \geqslant m \geqslant 0$). Recall from (4.5) that

$$\operatorname{Im} R_{-1,*} = \operatorname{Im} P_{-1,*}, \quad \operatorname{Ker} R_{-1,*} = \operatorname{Ker} P_{-1,*} \oplus W_{-1,*}. \tag{4.10}$$

Note that $U(0,-1)^*(X_{0,*}) \subset X_{-1,*} \subset \operatorname{Im} P_{-1,*}$ by Lemma 2.1(iii) and (3.21). Also, $U(0,-1)^*(\operatorname{Im} P_{0,*}) \subset \operatorname{Im} P_{-1,*}$ as in (4.4). Indeed, if $\xi = P_{0,*}\xi$ then $U(0,-1)^*\xi \in \operatorname{Im} P_{-1,*}$ by (3.18). Thus, we have $U(0,-1)^*(\operatorname{Im} R_{0,*}) \subset \operatorname{Im} R_{-1,*}$.

To prove $U(0,-1)^*(\operatorname{Ker} R_{0,*}) \subset \operatorname{Ker} R_{-1,*}$, we first remark (cf. (4.2)) that

$$U(0,-1)^* : \tilde{X}_{0,*} \to \operatorname{Ker} P_{-1,*} \text{ is an isomorphism.} \tag{4.11}$$

The proof of (4.11) is identical to the proof of (4.2) and uses the reduced node operator (3.20). Lemma 3.6(iii), (iv) implies that $U(0,-1)^* : Y_{0,*} \to W_{-1,*}$ is an isomorphism. Thus, by (4.9)–(4.11) we conclude that $U(0,-1)^*$: $\operatorname{Ker} R_{0,*} \to \operatorname{Ker} R_{-1,*}$ is an isomorphism. So, $U(0,-1)^* R_{0,*} = R_{-1,*} U(0,-1)^*$. Estimates (4.7)–(4.8) for $0 \geqslant n \geqslant m$ (with, generally, new M) follow from the estimates for $0 > n \geqslant m$ that have been previously proved in Proposition 3.4 and Lemma 3.5.

To finish the proof of Theorem 1.6 for $n \leqslant 0$, we denote $P_n^- = (R_{n,*})^*$, $n \leqslant 0$, and observe that $\operatorname{Im} P_n^- = \operatorname{Im}(R_{n,*})^* = (\operatorname{Ker} R_{n,*})^\perp = (\operatorname{Im} R_{n,*})^*$, and

$$\operatorname{Ker} P_n^- = \operatorname{Ker}(R_{n,*})^* = (\operatorname{Im} R_{n,*})^\perp = (\operatorname{Ker} R_{n,*})^*. \tag{4.12}$$

Passing to the adjoint operators in (4.7) and (4.8), we have for $0 \geqslant n \geqslant m$:

$$||U(n,m)|_{\operatorname{Im} P_m^-}|| \leqslant M e^{-\alpha(n-m)}, \quad ||(U(n,m)|_{\operatorname{Ker} P_m^-})^{-1}|| \leqslant M e^{-\alpha(n-m)}, \tag{4.13}$$

and Theorem 1.6 for $n \leqslant 0$ is proved. □

The next statement shows that the dimension of the kernel and cokernel of D is, in fact, equal to the dimension of the arbitrary fiber, cf. Lemma 2.1(i).

Corollary 4.1. *If D is Fredholm, then for each $n \in \mathbb{Z}$ we have* $\dim X_n = \dim \operatorname{Ker} D$ *and* $\dim X_{n,*} = \dim \operatorname{Ker} D^*$.

Proof. Fix $x \in X_n$, and let $x_k = U(k,n)x$ for $k \geqslant n$. By Lemma 2.1(ii), $x_k \in X_k$. Using (4.1) and Lemma 3.2, for $k > \max\{n, 0\}$ we have $x_k \in \operatorname{Im} P_k^+$. Thus, $||x_k|| \leqslant c e^{-\alpha k}$ for $k \geqslant 0$. If $k < n$ then by Lemma 2.1(ii) there exists a unique $x_k \in X_k$ such that $x = U(n,k)x_k$. Using (4.5) and (4.12), for $k < \min\{0, n\}$ we have $\operatorname{Ker} P_k^- = (\operatorname{Im} R_{k,*})^\perp = (\operatorname{Im} P_{k,*})^\perp \supset X_k$ since $\operatorname{Im} P_{k,*} \subset X_k^\perp$ in Proposition 3.4. Thus, $||x|| = ||U(n,k)x_k|| \geqslant c e^{-\alpha k} ||x_k||$ or $||x_k|| \leqslant c e^{\alpha k}$ for $k < 0$. Therefore, starting with an $x \in X_n$, we obtain an exponentially decaying sequence $(x_k)_{k \in \mathbb{Z}}$ (as $|k| \to \infty$) such that $x_k = U(k,m)x_m$ for all $k \geqslant m$ in $\mathbb{Z}$. Thus, $(x_k)_{k \in \mathbb{Z}} \in \operatorname{Ker} D$, and we can consider a well-defined and injective linear map $j_n : X_n \to \operatorname{Ker} D : x \mapsto (x_k)_{k \in \mathbb{Z}}$. It is surjective by the definition of X_n. Thus, X_n and $\operatorname{Ker} D$ are isomorphic. Similarly, $X_{n,*}$ is isomorphic to $\operatorname{Ker} D^*$. □

5. Proof of Theorems 1.1 and 1.2

In this section, in Proposition 5.2 we show that if D is Fredholm then the discrete node operator $N(n,m)$, $n \geqslant m$ in $\mathbb{Z}$, is Fredholm, and that $\operatorname{ind} D = \operatorname{ind} N(n,m)$.

Thus, Theorem 1.6 and Proposition 5.2 in combination with Theorem 1.4 yield implication (1.3) $\Rightarrow$ (i) and (ii) in Theorem 1.1. Finally, to complete the proofs of Theorems 1.1 and 1.2, we show that (i) and (ii) in Theorem 1.1 imply (1.3).

Consider two families of projections, $\{P_n^-\}_{n\leqslant 0}$ and $\{P_n^+\}_{n\geqslant 0}$. For $n\geqslant 0\geqslant m$ we define the discrete node operator $N(n,m)$ as follows:

$$N(n,m) := (I - P_n^+)U(n,m)|_{\operatorname{Ker} P_m^-} : \operatorname{Ker} P_m^- \to \operatorname{Ker} P_n^+.$$

Note that $N(0,0) = (I - P_0^+)|_{\operatorname{Ker} P_0^-}$ acts from $\operatorname{Ker} P_0^-$ to $\operatorname{Ker} P_0^+$, and $N(0,0) = (I - P_0^+)(I - P_0^-)|_{\operatorname{Ker} P_0^-}$. First, we reformulate the fact that $N(0,0)$ is Fredholm in terms of the associated Fredholm pair of subspaces.

Lemma 5.1. *If (P_0^+, P_0^-) is a pair of projections on X, then the node operator $N(0,0) = (I - P_0^+)|_{\operatorname{Ker} P_0^-}$ is a Fredholm operator from $\operatorname{Ker} P_0^-$ to $\operatorname{Ker} P_0^+$ if and only if the pair of subspaces $\operatorname{Ker} P_0^-$ and $\operatorname{Im} P_0^+$ is Fredholm in X. Moreover, $\dim \operatorname{Ker} N(0,0) = \alpha(\operatorname{Ker} P_0^-, \operatorname{Im} P_0^+)$, $\operatorname{codim} \operatorname{Im} N(0,0) = \beta(\operatorname{Ker} P_0^-, \operatorname{Im} P_0^+)$, and $\operatorname{ind} N(0,0) = \operatorname{ind}(\operatorname{Ker} P_0^-, \operatorname{Im} P_0^+)$.*

Proof. By the definition of $N(0,0)$ we have $\operatorname{Ker} N(0,0) = \operatorname{Ker} P_0^- \cap \operatorname{Im} P_0^+$. We claim that $\operatorname{Im} N(0,0) \oplus \operatorname{Im} P_0^+ = \operatorname{Ker} P_0^- + \operatorname{Im} P_0^+$. Indeed, if $x \in \operatorname{Ker} P_0^-$ then $y = N(0,0)x = x - P_0^+ x \in \operatorname{Ker} P_0^- + \operatorname{Im} P_0^+$, and the inclusion "$\subset$" holds. To prove the inclusion "$\supset$", take a $z = x + y$ with $x \in \operatorname{Ker} P_0^-$ and $y \in \operatorname{Im} P_0^+$. Then $(I - P_0^+)z = (I - P_0^+)x \in \operatorname{Im} N(0,0)$, and $z = (I - P_0^+)z + P_0^+ z \in \operatorname{Im} N(0,0) \oplus \operatorname{Im} P_0^+$. Using the claim, $\operatorname{Im} N(0,0)$ is a closed subspace in $\operatorname{Ker} P_0^+$ if and only if $\operatorname{Ker} P_0^- + \operatorname{Im} P_0^+$ is a closed subspace in X, and $\dim(\operatorname{Ker} P_0^+/\operatorname{Im} N(0,0)) = \dim(X/(\operatorname{Im} N(0,0) \oplus \operatorname{Im} P_0^+) = \beta(\operatorname{Ker} P_0^-, \operatorname{Im} P_0^+)$ for the quotient spaces. $\square$

Proposition 5.2. *If D is Fredholm on $\ell_p(\mathbb{Z};X)$, $p \in [1,\infty)$, or on $c_0(\mathbb{Z};X)$, then the discrete node operator $N(n,m)$, $n \geqslant 0 \geqslant m$, is Fredholm. Moreover, $\dim \operatorname{Ker} D = \dim \operatorname{Ker} N(n,m)$, $\operatorname{codim} \operatorname{Im} D = \operatorname{codim} \operatorname{Im} N(n,m)$, and $\operatorname{ind} D = \operatorname{ind} N(n,m)$.*

Proof. Consider the dichotomies $\{P_n^+\}_{n\geqslant 0}$ and $\{P_n^-\}_{n\leqslant 0}$ for $\{U(n,m)\}_{n\geqslant m}$, obtained in Theorem 1.6. Note that $N(n,m) = N(n,0)N(0,0)N(0,m)$, $n \geqslant 0 \geqslant m$, and that operators $N(n,0)$, $n>0$, and $N(0,m)$, $0>m$, are invertible. Thus, it suffices to prove that $N(0,0)$ is Fredholm and $\operatorname{ind} N(0,0) = \operatorname{ind} D (= \dim X_0 - \dim X_{0,*})$, see Corollary 4.1. We know that $\operatorname{Im} D$ is closed, and want to derive that $\operatorname{Im} N(0,0)$ is closed. First, we claim that if $y = (I - P_0^+)x$, $x \in \operatorname{Ker} P_0^-$, then $y \otimes \mathbf{e}_0 \in \operatorname{Im} D$. Indeed, define $x_n = (U(0,n)|_{\operatorname{Ker} P_n^-})^{-1} x$ for $n<0$ and $x_n = U(n,0)P_0^+ x$ for $n \geqslant 0$. Then for $n<0$ we have $x_n - U(n,n-1)x_{n-1} = (U(0,n)|_{\operatorname{Ker} P_0^-})^{-1} x - U(n,n-1)(U(0,n-1)|_{\operatorname{Ker} P_{n-1}^-})^{-1} x = 0$. Similarly, for $n>0$ we have $x_n - U(n,n-1)x_{n-1} =$

$U(n,0)P_0^+x - U(n,0)P_0^+x = 0$. For $n = 0$ we have

$$\begin{aligned} x_0 - U(0,-1)x_{-1} &= P_0^+x - U(0,-1)(U(0,-1)|_{\operatorname{Ker} P_{-1}^-})^{-1}x \\ &= P_0^+x - (I - P_0^-)x = P_0^+x - x = -y, \end{aligned}$$

where we have used that $x \in \operatorname{Ker} P_0^-$. Thus, $y \otimes \mathbf{e}_0 \in \operatorname{Im} D$ as claimed. Second, we claim that if $y \otimes \mathbf{e}_0 \in \operatorname{Im} D$ and $y \in \operatorname{Ker} P_0^+$, then $y \in \operatorname{Im} N(0,0)$. Indeed, for some $\mathbf{x} \in \ell_p(\mathbb{Z}; X)$ we have $D\mathbf{x} = y \otimes \mathbf{e}_0$. Thus $0 = x_n - U(n,0)x_0$ for $n > 0$. This implies $x_0 \in \operatorname{Im} P_0^+$. Also, $0 = x_{-1} - U(-1,n)x_n$ for $n \leqslant -1$. Therefore $x_{-1} \in \operatorname{Ker} P_{-1}^-$ and $U(0,-1)x_{-1} \in \operatorname{Ker} P_0^-$. Finally, $y = x_0 - U(0,-1)x_{-1}$ yields that $y = (I - P_0^+)y = (I - P_0^+)x_0 - (I - P_0^+)U(0,-1)x_{-1} = -(I - P_0^+)U(0,-1)x_{-1} \in \operatorname{Im} N(0,0)$ since $x_0 \in \operatorname{Im} P_0^+$ and $U(0,-1)x_{-1} \in \operatorname{Ker} P_0^-$, and the second claim is proved. Now assume $y = \lim_{j\to\infty} y^{(j)}$, where $y^{(j)} \in \operatorname{Im} N(0,0)$. By the first claim $y^{(j)} \otimes \mathbf{e}_0 \in \operatorname{Im} D, j \in \mathbb{N}$. Since $\operatorname{Im} D$ is closed, $y \otimes \mathbf{e}_0 = \lim_{j\to\infty} y^{(j)} \otimes \mathbf{e}_0 \in \operatorname{Im} D$. Since $\operatorname{Im} N(0,0) \subset \operatorname{Ker} P_0^+$, we also have $y \in \operatorname{Ker} P_0^+$. By the second claim $y \in \operatorname{Im} N(0,0)$, and thus $\operatorname{Im} N(0,0)$ is closed.

Next, we prove the formulas for the defect numbers. We have $\operatorname{Ker} N(0,0) = \operatorname{Ker} P_0^- \cap \operatorname{Im} P_0^+$. Thus, if $x \in \operatorname{Ker} N(0,0)$ then $\|x_n\| \leqslant ce^{-\alpha n}$, for $x_n = U(n,0)x$, $n \geqslant 0$, since $x \in \operatorname{Im} P_0^+$. Also, $\|x_n\| \leqslant ce^{\alpha n}$, $n < 0$, for the sequence $(x_n)_{n<0}$ such that $x = U(0,n)x_n$, $n < 0$, since $x \in \operatorname{Ker} P_0^-$. Thus, with this choice of x_n we have $x_n = U(n,m)x_m$ for all $n \geqslant m$, and $(x_n)_{n\in\mathbb{Z}} \in \operatorname{Ker} D$. Thus, $x \in X_0$. On the other hand,

$$\operatorname{Ker} P_0^- = (\operatorname{Im} R_{0,*})^\perp = (\operatorname{Im} P_{0,*} \oplus X_{0,*})^\perp = (\operatorname{Im} P_{0,*})^\perp \cap (X_{0,*})^\perp$$

by (4.12) and (4.9). Since $X_0 \subset \operatorname{Im} P_0^+ \subset X_{0,*}^\perp$ by (4.3) and $\operatorname{Im} P_{0,*} \subset X_0^\perp$ by Proposition 3.4, we have $\operatorname{Ker} N(0,0) = \operatorname{Im} P_0^+ \cap [X_{0,*}^\perp \cap (\operatorname{Im} P_{0,*})^\perp] = \operatorname{Im} P_0^+ \cap (\operatorname{Im} P_{0,*})^\perp \supset X_0$. So, $\operatorname{Ker} N(0,0) = X_0$, and $\dim \operatorname{Ker} N(0,0) = \dim X_0$. Further, $N(0,0)^* = (I - P_0^-)^*(I - P_0^+)^*$ is an operator acting from $(\operatorname{Ker} P_0^+)^* = \operatorname{Ker}(P_0^+)^*$ to $\operatorname{Ker}(P_0^-)^* = (\operatorname{Ker} P_0^-)^*$, and $\operatorname{Ker} N(0,0)^* = \operatorname{Im}(P_0^-)^* \cap \operatorname{Ker}(P_0^+)^*$. A similar argument yields $\dim N(0,0)^* = X_{0,*}$. $\square$

Proof of Theorems 1.1 and 1.2. Assume $\mathbf{G}$ is Fredholm. Then D is Fredholm by Theorem 1.4. By Theorem 1.6 there exist discrete dichotomies on $\mathbb{Z}_+$ and $\mathbb{Z}_-$. By Lemma 1.5, there exist dichotomies $\{P_t^+\}_{t\geqslant 0}$ and $\{P_t^-\}_{t\leqslant 0}$. This proves (i′) in Theorem 1.2 and, therefore, (i) in Theorem 1.1 for $a = 0 = b$. By Proposition 5.2 we also have that $N(0,0)$ is Fredholm, and, using formulas for the defect numbers and index from Theorem 1.4, we derive (ii′) in Theorem 1.2 and, by Lemma 5.1, (ii) in Theorem 1.1 for $a = 0 = b$, and the required formulas for the defect numbers and the index. It remains to prove that (i) and (ii) in Theorem 1.1 imply (1.3), see [9, Theorem 4], and also [7, Theorem 8] for the proof in the case when $a = -1$ and $b = 0$. We will present a proof, different form [7], as well as from the corresponding proofs in

[3,12,28,36,38,46] given in particular cases. Our proof is based on the following abstract fact from [29, p. 23].

Lemma 5.3. *Assume that a bounded linear operator A acting on a direct sum $\mathcal{X}_1 \oplus \mathcal{X}_2$ of two Banach spaces has the following triangular representation*:

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \text{ where } A_{11} \in \mathcal{L}(\mathcal{X}_1),\ A_{21} \in \mathcal{L}(\mathcal{X}_1, \mathcal{X}_2),\ A_{22} \in \mathcal{L}(\mathcal{X}_2). \tag{5.1}$$

Then A is Fredholm if and only if the following assertions hold:

(i) $\operatorname{Im} A_{11}$ *is closed, and* $\operatorname{codim} \operatorname{Im} A_{11} < \infty$;
(ii) $\operatorname{Im} A_{22}$ *is closed, and* $\dim \operatorname{Ker} A_{22} < \infty$;
(iii) *if* $\mathcal{L}_1 := \{x \in \mathcal{X}_1 : x \in \operatorname{Ker} A_{11}$ *and* $A_{21}x \in \operatorname{Im} A_{22}\}$ *then* $\dim \mathcal{L}_1$ *is finite*;
(iv) *if* $\mathcal{L}_2 := \operatorname{Im} A_{22} + A_{21}(\operatorname{Ker} A_{11})$ *then* $\operatorname{codim} \mathcal{L}_2$ *in* $\mathcal{X}_2$ *is finite.*

If (i)–(iv) *holds, then* $\dim \operatorname{Ker} A = \dim \operatorname{Ker} A_{22} + \dim \mathcal{L}_1$ *and* $\operatorname{codim} \operatorname{Im} A = \operatorname{codim} \operatorname{Im} A_{11} + \operatorname{codim} \mathcal{L}_2$.

By Theorem 1.4, it suffices to prove that D is Fredholm provided (i) and (ii) in Theorem 1.1 hold. We will present the proof for the ℓ_p-case, the c_0-case is similar. Passing to $[a]-1$ and $[b]+1$, if needed, where $[\cdot]$ is the integer part, we may assume that: (1) $a, b \in \mathbb{Z}$ in Theorem 1.1; (2) the discrete evolution family $\{U(n,m)\}_{n \geqslant m}$, $n, m \in \mathbb{Z}$, has dichotomies $\{P_n^-\}_{n \leqslant a}$ and $\{P_n^+\}_{n \geqslant b}$; and (3) the discrete node operator $N(b,a) = (I - P_b^+)U(b,a)|_{\operatorname{Ker} P_a^-}$ is a Fredholm operator from $\operatorname{Ker} P_a^-$ to $\operatorname{Ker} P_b^+$. First, for $A = D$ consider representation (5.1) for $\ell_p(\mathbb{Z}; X) = \mathcal{X}_1 \oplus \mathcal{X}_2$ with $\mathcal{X}_1 = \ell_p(\mathbb{Z} \cap (-\infty, b]; X)$ and $\mathcal{X}_2 = \ell_p(\mathbb{Z} \cap [b+1, \infty); X)$. Then $A_{11} = D_b^-$, where $D_b^- = D|_{\ell_p(\mathbb{Z} \cap (-\infty, b]; X)}$, $A_{22} = D_b^+$, where $D_b^+ : (x_n)_{n \geqslant b+1} \mapsto (x_{b+1}, x_{b+2} - U(b+2, b+1)x_{b+1}, \ldots)$, and $A_{21} = D_b^{\pm}$, where $D_b^{\pm} : (x_n)_{n \leqslant b} \mapsto (-U(b+1,b)x_b, 0, \ldots)$. Therefore,

$$\mathcal{L}_1 = \{(x_n)_{n \leqslant b} : (x_n)_{n \leqslant b} \in \operatorname{Ker} D_b^- \text{ and } (-U(b+1,b)x_b, 0, \ldots) \in \operatorname{Im} D_b^+\}, \tag{5.2}$$

$$\begin{aligned} \mathcal{L}_2 = \{&(x_n)_{n \geqslant b+1} + (-U(b+1,b)x_b, 0, \ldots) : (x_n)_{n \geqslant b+1} \in \operatorname{Im} D_b^+ \text{ and} \\ &(x_n)_{n \leqslant b} \in \operatorname{Ker} D_b^-\}. \end{aligned} \tag{5.3}$$

We will need a version of [4, Corollary 1]. For a sequence $(x_n)_{n \geqslant b+2}$ denote

$$x'_{b+1} = -\sum_{k=1}^{\infty} (U(b+1+k, b+1)|_{\operatorname{Ker} P_{b+1}^+})^{-1} (I - P_{b+1+k}^+) x_{b+1+k}. \tag{5.4}$$

The series in (5.4) converges by the unstable dichotomy estimate.

Lemma 5.4. *The operator D_b^+ is left-invertible on $\ell_p(\mathbb{Z}\cap[b+1,\infty);X)$, and*

$$\operatorname{Im} D_b^+ = \{(x_n)_{n\geqslant b+1} : (I-P_{b+1}^+)x_{b+1} = x'_{b+1}\}. \tag{5.5}$$

Proof. To construct $(D_b^+)^{-1}$, the left inverse for D_b^+, note that $D_b^+ = I - T_b^+$, where $T_b^+ : (x_n)_{n\geqslant b+1} \mapsto (0, U(b+2,b+1)x_{b+1}, \ldots)$. Decompose $T_b^+ = T_{b,s}^+ \oplus T_{b,u}^+$, where $T_{b,s}^+$, respectively, $T_{b,u}^+$, is the restriction of T_b^+ on the subspace of sequences $(x_n)_{n\geqslant b+1}$ from $\ell_p(\mathbb{Z}\cap[b+1,\infty);X)$ such that $x_n\in\operatorname{Im} P_n^+$, respectively, $x_n\in\operatorname{Ker} P_n^+$, $n\geqslant b+1$. Then $T_{b,u}^+$ is left invertible with the left inverse $(T_{b,u}^+)^{-1} : (x_n)_{n\geqslant b+1}\mapsto (U(n+1,n)|_{\operatorname{Ker} P_n^+})^{-1}x_{n+1})_{n\geqslant b+1}$. By the dichotomy assumption, $\operatorname{sprad}(T_{b,s}^+)<1$ and $\operatorname{sprad}((T_{b,u}^+)^{-1})<1$, and thus $(D_b^+)^{-1} = \sum_{k=0}^{\infty}(T_{b,s}^+)^k - \sum_{k=1}^{\infty}(T_{b,u}^+)^{-k}$. A calculation shows that $D_b^+(D_b^+)^{-1}$ maps a sequence $(x_n)_{n\geqslant b+1}$ to the sequence $(P_{b+1}^+x_{b+1} + x'_{b+1}, x_{b+2}, \ldots)$, see (5.4). Since $\operatorname{Im} D_b^+ = \operatorname{Im}(D_b^+(D_b^+)^{-1})$, we obtain (5.5). $\square$

Using the decomposition $\ell_p(\mathbb{Z}\cap(-\infty,b];X) = \mathscr{X}_1\oplus\mathscr{X}_2$, where $\mathscr{X}_1 = \ell_p(\mathbb{Z}\cap(-\infty,a-1];X)$ and $\mathscr{X}_2 = \ell_p(\mathbb{Z}\cap[a,b];X)$, consider representation (5.1) for $A = D_b^-$. We now have $A_{11} = D_{a-1}^- = D|_{\ell_p(\mathbb{Z}\cap(-\infty,a-1];X)}$, and also $A_{22} = D_{a,b}$, where $D_{a,b} : (x_n)_{a\leqslant n\leqslant b}\mapsto (x_a, x_{a+1} - U(a+1,a)x_a, \ldots, x_b - U(b,b-1)x_{b-1})$. In the representation $\ell_p(\mathbb{Z}\cap[a,b];X) = X\oplus\cdots\oplus X$ ($(b-a)$-times) the operator $D_{a,b}$ is lower-triangular with identities on the diagonal and, hence, invertible. Using dichotomy $\{P_n^-\}_{n\leqslant a-1}$, similarly to the proof of Lemma 5.4, we conclude that D_{a-1}^- is right-invertible. Since D_b^- is lower triangular with the diagonal blocks D_{a-1}^- and $D_{a,b}$, it follows that D_b^- is right-invertible. This and Lemma 5.4 imply that for the triangular representation (5.1) of D both assertions (i) and (ii) hold. Thus, to conclude that D is Fredholm, it remains to prove that $\dim\mathscr{L}_1<\infty$ and $\operatorname{codim}\mathscr{L}_2<\infty$ for $\mathscr{L}_1$ and $\mathscr{L}_2$ in (5.2)–(5.3). As soon as this is proved, $\dim\operatorname{Ker} D = \dim\mathscr{L}_1$ and $\operatorname{codim}\operatorname{Im} D = \operatorname{codim}\mathscr{L}_2$.

To handle $\mathscr{L}_1$, remark that $(-U(b+1,b)x_b, 0, \ldots)\in\operatorname{Im} D_b^+$ if and only if there exists a $(y_n)_{n\geqslant b+1}\in\ell_p(\mathbb{Z}\cap[b+1,\infty);X)$ such that $y_n = -U(n,b)x_b$, $n\geqslant b+1$. Using the dichotomy $\{P_n^+\}_{n\geqslant b}$, this is equivalent to $x_b\in\operatorname{Im} P_b^+$. On the other hand, $(x_n)_{n\leqslant b}\in\operatorname{Ker} D_b^-$ means that $x_n = U(n,m)x_m$ for all $m\leqslant n\leqslant b$. In particular, $x_b = U(b,a)x_a$, and $x_a = U(a,n)x_n$ for all $n\leqslant a$. Using the dichotomy $\{P_n^-\}_{n\leqslant a}$, we infer $x_a\in\operatorname{Ker} P_a^-$. Thus,

$$\dim\mathscr{L}_1 = \dim\{x\in\operatorname{Ker} P_a^- : U(b,a)x\in\operatorname{Im} P_b^+\} = \dim\operatorname{Ker} N(b,a)<\infty.$$

To handle $\mathscr{L}_2$, let Z denote any direct complement of $\operatorname{Im} N(b,a)$, such that $\operatorname{Ker} P_b^+ = \operatorname{Im} N(b,a)\oplus Z$, and let $[(x_n)_{n\geqslant b+1}]_{\mathscr{L}_2}$ for any $(x_n)_{n\geqslant b+1}\in\ell_p(\mathbb{Z}\cap[b+1,\infty);X)$ denote the equivalence class in the quotient space $\ell_p(\mathbb{Z}\cap[b+1,\infty);X)/\mathscr{L}_2$. By Lemma 5.4 we have $(P_n^+x_n)_{n\geqslant b+1}\in\operatorname{Im} D_b^+\subset\mathscr{L}_2$. Thus, $[(x_n)_{n\geqslant b+1}]_{\mathscr{L}_2} = [((I-P_n^+)x_n)_{n\geqslant b+1}]_{\mathscr{L}_2}$. Using (5.2), by Lemma 5.4 we infer

$(x'_{b+1}, (I - P^+_{b+2})x_{b+2}, \dots) \in \operatorname{Im} D^+_b \subset \mathscr{L}_2$, so, $[(x_n)_{n \geqslant b+1}]_{\mathscr{L}_2} = [(y_{b+1}, 0, \dots)]_{\mathscr{L}_2}$, where we denote $y_{b+1} = (I - P^+_{b+1})x_{b+1} - x'_{b+1}$. Note that $y_{b+1} \in \operatorname{Ker} P^+_{b+1}$, and find the unique $y_b \in \operatorname{Ker} P^+_b$ such that $y_{b+1} = U(b+1, b)|_{\operatorname{Ker} P^+_b} y_b$. Using the decomposition $\operatorname{Ker} P^+_b = \operatorname{Im} N(b,a) \oplus Z$, find the unique representation $y_b = y + z$, where $y \in \operatorname{Im} N(b,a)$ and $z \in Z$. Since $y \in \operatorname{Im} N(b,a)$, there is an $x_a \in \operatorname{Ker} P^-_a$ such that $y = U(b,a)x_a$. Using the dichotomy $\{P^-_n\}_{n \leqslant a}$, set $x_n = (U(a,n)|_{\operatorname{Ker} P^-_n})^{-1} x_a$ for $n \leqslant a$. Also, define $x_n = U(n,a)x_a$ for $n \in [a,b]$. Then $(x_n)_{n \leqslant b} \in \ell_p(\mathbb{Z} \cap (-\infty, b]; X)$ and $x_n = U(n,m)x_m$ for all $m \leqslant n \leqslant b$. Thus, $(x_n)_{n \leqslant b} \in \operatorname{Ker} D^-_b$. Also, $y = x_b$. By (5.3) then $[(-U(b+1,b)y, 0, \dots)]_{\mathscr{L}_2} = [(U(b+1,b)z, 0, \dots)]_{\mathscr{L}_2}$. As a result, we have a well-defined map $j : \mathbf{x} = [(x_n)_{n \geqslant b+1}]_{\mathscr{L}_2} \mapsto z$ from $\ell_p(\mathbb{Z} \cap [b+1, \infty))/\mathscr{L}_2$ to $Z \cong \operatorname{Ker} P^+_b / \operatorname{Im} N(b,a)$ such that $[(x_n)_{n \geqslant b+1}]_{\mathscr{L}_2} = [(U(b+1,b)z, 0, \dots)]_{\mathscr{L}_2}$ with $j\mathbf{x} = z$. It follows that j is injective. It is surjective, since if $z \in Z$ then $\mathbf{x} = [(U(b+1,b)z, 0, \dots)]_{\mathscr{L}_2}$ satisfies $j\mathbf{x} = z$. $\square$

6. Differential and difference operators

In this section we prove Theorem 1.4 and Lemma 1.5. The proof is given for the case of $L_p(\mathbb{R}; X)$, $p \in [1, \infty)$, the case of $C_0(\mathbb{R}; X)$ is similar. Fix a continuous 1-periodic function $\alpha : \mathbb{R} \to \mathbb{R}$ such that $\alpha(0) = \alpha(1) = 0$ and $\int_0^1 \alpha(s)ds = 1$, and recall notation $\mathbf{x} = (x_n)_{n \in \mathbb{Z}}$. Define bounded linear operators $R : L_p(\mathbb{R}; X) \to \ell_p(\mathbb{Z}; X)$ and $S : \ell_p(\mathbb{Z}; X) \to L_p(\mathbb{R}; X)$ as follows:

$$Rf = \left(-\int_{n-1}^{n} U(n,s)f(s)\,ds \right)_{n \in \mathbb{Z}}, \quad (S\mathbf{x})(t) = \alpha(t)U(t,n)x_n, \quad t \in [n, n+1].$$

Lemma 6.1. (i) *If* $\mathbf{y} = D\mathbf{x}$ *then* $\mathbf{G}u = S\mathbf{y}$ *for some* $u \in \operatorname{dom} \mathbf{G}$;

(ii) *if* $S\mathbf{y} = \mathbf{G}u$ *for some* $u \in \operatorname{dom} \mathbf{G}$ *then* $\mathbf{y} = D\mathbf{x}$ *for some* $\mathbf{x} \in \ell_p$;

(iii) *if* $f = \mathbf{G}u$ *for some* $u \in \operatorname{dom} \mathbf{G}$, then $Rf = D\mathbf{x}$ *with* $\mathbf{x} = (u(n))_{n \in \mathbb{Z}}$;

(iv) *if* $Rf = D\mathbf{x}$ *for some* $\mathbf{x} \in \ell_p$, *then* $f = \mathbf{G}u$ *for some* $u \in \operatorname{dom} \mathbf{G}$.

Proof. (i) Define $u(t) = U(t,n)(y_n - x_n) - \int_n^t U(t,s)S\mathbf{y}(s)\,ds$ for $t \in [n, n+1]$. A direct but tedious calculation similar to [15, p. 117] shows that $u \in L_p(\mathbb{R}; X) \cap C_0(\mathbb{R}; X)$ and satisfies (1.4) with $f = S\mathbf{y}$. Thus $\mathbf{G}u = S\mathbf{y}$.

(ii) For $u \in L_p(\mathbb{R}; X) \cap C_0(\mathbb{R}; X)$ satisfying (1.4) with $f = S\mathbf{y}$ we have for $t = n+1$ and $\tau = n$:

$$\begin{aligned} u(n+1) &= U(n+1,n)u(n) - \int_n^{n+1} U(n+1,s)\alpha(s)U(s,n)y_n\,ds \\ &= U(n+1,n)u(n) - U(n+1,n)y_n, \quad n \in \mathbb{Z}. \end{aligned}$$

Thus, $\mathbf{y} = D(y_n - u(n))_{n \in \mathbb{Z}}$. The proof of $(u(n))_{n \in z} \in \ell_p$ is similar to [4].

(iii) Since u and f satisfy (1.4), letting $t = n$ and $\tau = n-1$, we have that $-\int_{n-1}^{n} U(n,s)f(s)\,ds = u(n) - U(n,n-1)u(n-1)$, $n \in \mathbb{Z}$. $-\int_{n-1}^{n} U(n,s)f(s)$, $ds = u(n) - U(n,n-1)u(n-1)$, $n \in \mathbb{Z}$. The proof of $(u(n))_{n\in\mathbb{Z}} \in \ell_p$ is similar to [4].

(iv) For $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$ such that $Rf = D\mathbf{x}$ define

$$u(t) = U(t,n)x_n - \int_n^t U(t,s)f(s)ds, \quad t \in [n, n+1], \ n \in \mathbb{Z}.$$

A calculation similar to [15, p. 117] again shows that $u \in L_p(\mathbb{R};X) \cap C_0(\mathbb{R};X)$, and that u and f satisfy (1.4). Thus, $\mathbf{G}u = f$.

We now claim that $\operatorname{Im}\mathbf{G}$ is closed if and only if $\operatorname{Im} D$ is closed. Assume that $\operatorname{Im} D$ is closed, and consider any sequence $f^{(k)} = \mathbf{G}u^{(k)} \in \operatorname{Im}\mathbf{G}$ such that $\lim_{k\to\infty} f^{(k)} = f$ in $L_p(\mathbb{R};X)$. Using Lemma 6.1(iii) we have $Rf^{(k)} = D(u^{(k)}(n))_{n\in\mathbb{Z}} \to Rf$, $k \to \infty$. Since $\operatorname{Im} D$ is closed, $Rf \in \operatorname{Im} D$ and thus $f \in \operatorname{Im}\mathbf{G}$ by Lemma 6.1(iv). Conversely, assume that $\operatorname{Im}\mathbf{G}$ is closed, and consider any sequence $\mathbf{y}^{(k)} = D\mathbf{x}^{(k)} \in \operatorname{Im} D$ such that $\lim_{k\to\infty} \mathbf{y}^{(k)} = \mathbf{y}$ in ℓ_p. Using Lemma 6.1(i), we have $S\mathbf{y}^{(k)} = \mathbf{G}u^{(k)} \to S\mathbf{y}$ for some $u^{(k)} \in \operatorname{dom}\mathbf{G}$. Since $\operatorname{Im}\mathbf{G}$ is closed, $S\mathbf{y} \in \operatorname{Im}\mathbf{G}$ and thus $\mathbf{y} \in \operatorname{Im} D$ by Lemma 6.1(ii). This proves the claim.

Define a linear map, B, by $(B\mathbf{x})(t) = U(t,n)x_n$, $t \in [n, n+1)$, $n \in \mathbb{Z}$, where $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$. According to (1.4), $u \in \operatorname{Ker}\mathbf{G}$ if and only if $u \in L_p(\mathbb{R};X) \cap C_0(\mathbb{R};X)$ and $u(t) = U(t,\tau)u(\tau)$ for all $t \geqslant \tau$ in $\mathbb{R}$. By (2.4), B is an injective map from $\operatorname{Ker} D$ to $\operatorname{Ker}\mathbf{G}$. If $u \in \operatorname{Ker}\mathbf{G}$ then $B(u(n))_{n\in\mathbb{Z}} = u$ shows that B is surjective. Thus, $\operatorname{Ker} D$ and $\operatorname{Ker}\mathbf{G}$ are isomorphic, and $\dim\operatorname{Ker}\mathbf{G} = \dim\operatorname{Ker} D$.

Finally, we show that if $\operatorname{Im}\mathbf{G}$ (equivalently, $\operatorname{Im} D$) is closed, then $\dim\hat{L}_p = \dim\hat{\ell}_p$ for the quotient spaces $\hat{L}_p \coloneqq \{[f] = \{f + g : g \in \operatorname{Im}\mathbf{G}\} : f \in L_p\}$ and $\hat{\ell}_p \coloneqq \{[\mathbf{y}] = \{\mathbf{y} + \mathbf{z} : \mathbf{z} \in \operatorname{Im} D\} : \mathbf{y} \in \ell_p\}$. Indeed, define the operator $\hat{R} : \hat{L}_p \to \hat{\ell}_p$, by the rule $\hat{R}[f] = [Rf]$. Since $g \in \operatorname{Im}\mathbf{G}$ implies $Rg \in \operatorname{Im} D$ by Lemma 6.1(iii), if $h = f + g \in [f]$, $g \in \operatorname{Im}\mathbf{G}$, then $Rh = Rf + Rg \in [Rf]$, and $\hat{R}$ is well-defined. If $\hat{R}[f] = 0$, then $Rf \in \operatorname{Im} D$ and, by Lemma 6.1(iv) we have $f \in \operatorname{Im}\mathbf{G}$ and thus $[f] = 0$. So, $\hat{R}$ is injective. Fix $\mathbf{y} = (y_n)_{n\in\mathbb{Z}} \in \ell_p$, and let $f = -S\mathbf{y}$. Then

$$(Rf)_n = \int_{n-1}^{n} U(n,s)\alpha(s)U(s,n-1)y_{n-1}\,ds = y_n - (D\mathbf{y})_n.$$

So, $\mathbf{y} = Rf + D\mathbf{y}$. Then $[\mathbf{y}] = [Rf] = \hat{R}[f]$, and $\hat{R}$ is surjective. Thus, $\hat{L}_p$ and $\hat{\ell}_p$ are isomorphic. $\square$

Proof of Lemma 1.5. We give the proof of the "only if" part for $\mathbb{R}_+$, arguments for $\mathbb{R}_-$ are similar. Due to the dichotomy estimates for the family $\{U(n,m)\}_{n\geqslant m\geqslant 0}$, we claim that it suffices to construct $\{P_t^+\}_{t\geqslant 0}$ such that $U(t,\tau)P_\tau^+ = P_t^+U(t,\tau)$ and $U(t,\tau)|_{\operatorname{Ker} P_\tau^+} : \operatorname{Ker} P_\tau^+ \to \operatorname{Ker} P_t^+$ is an isomorphism for all $t \geqslant \tau \geqslant 0$. Indeed, assume that the claim is proved. Then the stable exponential dichotomy estimate for

$\{U(t,\tau)\}_{t\geqslant\tau\geqslant 0}$ follows directly from the stable dichotomy estimate for $\{U(n,m)\}_{n\geqslant m\geqslant 0}$ since $\sup_{0\leqslant t-\tau\leqslant 1}\|U(t,\tau)\|<\infty$. To obtain the unstable dichotomy estimate for $\{U(t,\tau)\}_{t\geqslant\tau\geqslant 0}$, note that if $n+1\geqslant t\geqslant n\geqslant m\geqslant\tau\geqslant m-1\geqslant 0$ then

$$(U(t,\tau)|_{\mathrm{Ker}\,P_\tau^+})^{-1}=(U(m,\tau)|_{\mathrm{Ker}\,P_\tau^+})^{-1}(U(n,m)|_{\mathrm{Ker}\,P_m^+})^{-1}(U(t,n)|_{\mathrm{Ker}\,P_n^+})^{-1}. \quad (6.1)$$

But $(U(t,n)|_{\mathrm{Ker}\,P_n^+})^{-1}=(U(n+1,n)|_{\mathrm{Ker}\,P_n^+})^{-1}(U(n+1,t)|_{\mathrm{Ker}\,P_t})$. Using the unstable dichotomy estimate for $\{U(n,m)\}_{n\geqslant m\geqslant 0}$, and the fact that $\sup\{\|U(n+1,t)\|: n\in\mathbb{Z}_+,\ t\in[n,n+1]\}<\infty$, we have that $\sup\{\|(U(t,n)|_{\mathrm{Ker}\,P_n^+})^{-1}\|: n\in\mathbb{Z}_+,\ t\in[n,n+1]\}<\infty$ and, similarly, that $\sup\{\|(U(m,\tau)|_{\mathrm{Ker}\,P_\tau^+})^{-1}\|: m\in\mathbb{Z}_+, m\geqslant 1, \tau\in[m-1,m]\}<\infty$. Now (6.1) implies the unstable dichotomy estimate for $\{U(t,\tau)\}_{t\geqslant\tau\geqslant 0}$. To prove the claim, fix $t_0\in\mathbb{R}$ so that $t_0\in[n,n+1)$ for some $n\in\mathbb{Z}_+$, and define subspaces $X_s(t_0)=\{x\in X: U(n+1,t_0)x\in\mathrm{Im}\,P_{n+1}^+\}$ and $X_u(t_0)=U(t_0,n)(\mathrm{Ker}\,P_n^+)$. Using the unstable dichotomy estimate for $\{U(n,m)\}_{n\geqslant m\geqslant 0}$, for each $x\in\mathrm{Ker}\,P_n^+$ we have $\|U(n+1,t_0)\|\|U(t_0,n)x\|\geqslant\|U(n+1,n)x\|\geqslant M^{-1}e^{\alpha}\|x\|$. Thus, $U(t_0,n):\mathrm{Ker}\,P_n^+\to X_u(t_0)$ is an isomorphism, and $X_u(t_0)$ is closed. Also, $U(t_1,t_0): X_u(t_0)\to X_u(t_1)$ is an isomorphism for all $t_1\geqslant t_0$ in $\mathbb{R}_+$. If $x\in X_s(t_0)\cap X_u(t_0)$, then $U(n+1,t_0)x\in\mathrm{Im}\,P_{n+1}^+$ and there is a $y\in\mathrm{Ker}\,P_n^+$ such that $x=U(t_0,n)y$. Then $U(n+1,n)y=U(n+1,t_0)x\in\mathrm{Im}\,P_{n+1}^+$. Thus, $U(n+1,n)y=0$ and $y=0$ since $U(n+1,n):\mathrm{Ker}\,P_n^+\to\mathrm{Ker}\,P_{n+1}^+$ is an isomorphism. Thus, $X_s(t_0)\cap X_u(t_0)=\{0\}$. To prove that $X=X_s(t_0)\oplus X_u(t_0)$, take an $x\in X$, and decompose $U(n+1,t_0)x=y_s+y_u$, $y_s\in\mathrm{Im}\,P_{n+1}^+$, $y_u\in\mathrm{Ker}\,P_{n+1}^+=X_u(n+1)$. Let x_u denote the unique vector in $X_u(t_0)$ such that $U(n+1,t_0)x_u=y_u$, and let $x_s=x-x_u$. Then $x_s\in X_s(t_0)$ since $U(n+1,t_0)x_s=y_s\in\mathrm{Im}\,P_{n+1}^+$. Projections P_t^+, $t\geqslant 0$, with $\mathrm{Im}\,P_t^+=X_s(t)$, $\mathrm{Ker}\,P_t^+=X_u(t)$ give the desired dichotomy. The proof of the "if" part of the lemma is straightforward. $\square$

7. Special cases

In this section we discuss several particular cases when the statements of Theorems 1.1 and 1.2 allow certain simplifications, and indicate classes of problems for which these theorems could be applied. We present the results only for $L_p=L_p(\mathbb{R};X)$, $p\in[1,\infty)$. In this section all differential equations $u'(t)=A(t)u(t)$ with, generally, unbounded operators $A(t)$, $t\in\mathbb{R}$, are assumed to be well-posed in the following $W_p^1(\mathbb{R};X)$-sense (cf. [48, p. 313]): (1) There exists a dense subset $\mathscr{D}\subset X$ such that $\mathrm{dom}\,A(t)=\mathscr{D}$ for all $t\in\mathbb{R}$; and (2) There exists a strongly continuous exponentially bounded evolution family $\{U(t,\tau)\}_{t\geqslant\tau}$, $t,\tau\in\mathbb{R}$, on X so that for all $\tau\in\mathbb{R}$ and each $x_\tau\in\mathscr{D}$ the function $u(t)=U(t,\tau)x_\tau$, defined for $t\geqslant\tau$, takes values in $\mathscr{D}$, belongs to the Sobolev space $W_p^1([\tau,T];X)$ for every $T\geqslant\tau$, and satisfies the differential equation $u'(t)=A(t)u(t)$ for almost all $t\in[\tau,T]$.

Mild and regular solutions: The operator $\mathbf{G}$, described in Lemma 1.3, is the generator of the evolution semigroup induced by the propagator $\{U(t,\tau)\}_{t\geqslant\tau}$ of the well posed differential equation $u'(t)=A(t)u(t)$, $t\in\mathbb{R}$. Therefore, $\mathbf{G}$ is a closed operator on $L_p(\mathbb{R};X)$, $p\in[1,\infty)$. Also, $u:\mathbb{R}\to X$ is a mild solution of the inhomogeneous equation $u'(t)=A(t)u(t)+f(t)$, $t\in\mathbb{R}$, for $f\in L_p(\mathbb{R};X)$, provided $u\in\operatorname{dom}\mathbf{G}$ and $\mathbf{G}u=f$. Consider the operator $G=-d/dt+A(t)$ with the domain $\operatorname{dom} G$ given in (1.2). We say that u is a regular solution of the inhomogeneous equation provided $u\in\operatorname{dom} G$ and $Gu=f$. Note that for many classes of equations (say, parabolic) mild solutions have additional regularity. If this is the case, one might expect that $\mathbf{G}=G$. The latter equality is indeed true provided, for instance, that the inhomogeneous equation $u'(t)=A(t)u(t)+f(t)$ has L_p-maximal regularity, a property established for a large variety of parabolic nonautonomous problems, see [30,48] for further references.

Recall that, by Chicone and Latushkin [15, Theorem 3.12] and Schnaubelt [48, Proposition 4.1], the set $\operatorname{dom} G$ from (1.2) is a core for $\mathbf{G}$. Thus, if G is closed then $\mathbf{G}=G$. As a result, we conclude that if G is a closed operator on $L_p(\mathbb{R};X)$, $p\in[1,\infty)$, then Theorems 1.1 and 1.2, and all other results of this paper, are valid if the operator $\mathbf{G}$ in their formulations is replaced by G. We will not go into discussion of the (quite delicate, see [48, Section (c)]) question when G is closed, but merely mention that $\mathbf{G}=G$ under the following simplest assumption:

$$A:\mathbb{R}\mapsto\mathscr{L}(X)\text{ is piecewise continuous and }\sup_{t\in\mathbb{R}}\|A(t)\|<\infty. \tag{7.1}$$

Indeed, in this case the propagator $\{U(t,\tau)\}_{t,\tau\in\mathbb{R}}$ is differentiable in $\mathscr{L}(X)$. Then $u\in W_p^1(\mathbb{R};X)$ is a regular solution of the inhomogeneous equation if and only if u is a mild solution of this equation. Therefore, $\mathbf{G}=G$ for the operator $G=-d/dt+A(t)$ with $\operatorname{dom} G=W_p^1(\mathbb{R};X)$, $p\in[1,\infty)$.

Compactness and node operators: In many cases studied in the literature the operator $\mathbf{G}$ (or G, defined in (1.1) with domain (1.2)) was proved to be Fredholm if and only if the corresponding evolution family (or the differential equation $u'(t)=A(t)u(t)$, $t\in\mathbb{R}$) has exponential dichotomies on $\mathbb{R}_+$ and $\mathbb{R}_-$, see, e.g., [12, Theorem 1.2; 21, Theorem 1.1; 28, Lemma 3.4; 36, Lemma 4.2; 37; 46, Theorem 2.6; 53, Theorem 1.3]. Thus, in these papers condition (ii′) in Theorem 1.2 or, equivalently, see Lemma 5.1, condition (ii) in Theorem 1.1 has been fulfilled automatically. A reason for this is explained in Lemma 7.3 below. Indeed, under the assumptions imposed in the above cited papers, or for the classes of the evolution families studied in these papers, the projectors $I-P_0^+$ and $I-P_0^-$ happened to be of finite rank (and thus compact), or their difference was compact. If, for instance, $U(t,\tau)$ are compact operators in X for all $t>\tau$ in $\mathbb{R}$, then the invertibility of their restrictions $U(t,\tau)|_{\operatorname{Ker} P_\tau}$ acting from $\operatorname{Ker} P_\tau$ to $\operatorname{Ker} P_t$ (see (ii) in the definition of the exponential dichotomy) implies that $\operatorname{Ker} P_\tau$ is finite dimensional. The more general α-*contractivity* condition on $U(t,\tau)$ also implies that $\operatorname{Ker} P_\tau$ is finite dimensional, see, e.g., [44, p. 21] and the literature cited therein. The following two examples, on the contrary, identify important autonomous equations $u'(t) = Au(t)$ for which *both* stable and unstable subspaces are infinite dimensional, see also [38,46].

Example 7.1 (Petrovskij correct systems). Let $\mathbf{p}(\xi) = [\mathbf{p}_{kj}(\xi)]_{k,j=1}^{K}$, $\xi \in \mathbb{R}^d$, $d \geqslant 1$, be a $(K \times K)$ matrix whose entries are complex-valued polynomials $\mathbf{p}_{kj}(\xi) = \sum_{|\alpha| \leqslant N_{kj}} a_\alpha \xi^\alpha$. Here we use the multiindex notation for $\alpha \in \mathbb{N}^d$, and $a_\alpha \in \mathbb{C}$ depend on k and j. In $L_2(\mathbb{R}^d; \mathbb{C}^K)$ the operator $A = \mathbf{p}(i\partial)$, $\partial = (\partial_1, \dots, \partial_d)$, $i^2 = -1$, is defined via Fourier transform, $A = \mathscr{F}^{-1}\mathbf{p}(\cdot)\mathscr{F}$, and is a general (matrix) constant coefficient operator with the symbol $\mathbf{p}$. We say that A is *Petrovskij correct* if for some $\omega \in \mathbb{R}$ the spectrum $\sigma(\mathbf{p}(\xi))$ of the matrix $\mathbf{p}(\xi)$ satisfies $\sigma(\mathbf{p}(\xi)) \subset \{z \in \mathbb{C} : \operatorname{Re} z \leqslant \omega\}$ for all $\xi \in \mathbb{R}^d$. If this is the case, then A generates a strongly continuous semigroup on $L_2(\mathbb{R}^d)$, where $\operatorname{dom} A$ is the Sobolev space of order $N = \max N_{kj}$. This semigroup is hyperbolic provided $\sigma(\mathbf{p}(\xi))$ is uniformly separated from $i\mathbb{R}$ for all $\xi \in \mathbb{R}^d$. Both stable and unstable spectral subspaces can be infinite dimensional. A "toy" (2×2) matrix first order example is $\mathbf{p}(\xi) = \operatorname{diag}[i\xi - a, i\xi + b]$, $\xi \in \mathbb{R}$, $a, b > 0$, where $\sigma(A) = (i\mathbb{R} - a) \cup (i\mathbb{R} + b)$. For a study of dichotomy of hyperbolic systems with constant and close to constant coefficients see [25,51] and the literature therein.

Example 7.2 (Schrödinger operators with periodic potentials). Consider on $X = L_2(\mathbb{R}; \mathbb{C})$ a Schrödinger operator $A = \frac{d^2}{dx^2} + V(x)$, $\operatorname{dom} A = W_2^2(\mathbb{R}; \mathbb{C})$, with a piecewise continuous real-valued periodic potential V. By Theorem XIII.90 from [40] we know that its spectrum $\sigma(A) = \bigcup_{n=1}^{\infty} [\alpha_n, \beta_n]$ for some $\beta_n \leqslant \alpha_{n+1}$, and $\sigma(A)$ is absolutely continuous; also, unless V is a constant, $\alpha_{n+1} \neq \beta_n$ for some n, that is, there are gaps in $\sigma(A)$ (e.g., $\alpha_{n+1} \neq \beta_n$ for all $n \in \mathbb{N}$ for the Mathieu potential $V(x) = \mu \cos x$, $\mu \neq 0$). Thus, if $0 \in (\beta_n, \alpha_{n+1})$ for some n then the equation $u'(t) = Au(t)$ has an exponential dichotomy on $\mathbb{R}$ with infinite dimensional stable and unstable subspaces.

Lemma 7.3. *If P_0^+ and P_0^- are projectors on a Banach space X, and $P_0^+ - P_0^-$ is a compact operator, then the node operator $N(0,0) = (I - P_0^+)|_{\operatorname{Ker} P_0^-} : \operatorname{Ker} P_0^- \to \operatorname{Ker} P_0^+$ is Fredholm.*

Proof. A (2×2) matrix representation (2.1) of the Fredholm operator $L = I - (P_0^+ - P_0^-)$ acting from $X = \operatorname{Im} P_0^- \oplus \operatorname{Ker} P_0^-$ to $X = \operatorname{Im} P_0^+ \oplus \operatorname{Ker} P_0^+$ has the form $L = \begin{bmatrix} P_0^+ P_0^- & 0 \\ 2(I - P_0^+)P_0^- & N(0,0) \end{bmatrix}$, where $N(0,0) = (I - P_0^+)(I - P_0^-) : \operatorname{Ker} P_0^- \to \operatorname{Ker} P_0^+$. By (ii) in Lemma 5.3, $\operatorname{Im} N(0,0)$ is closed and $\dim \operatorname{Ker} N(0,0) < \infty$. Passing to the adjoints, $N(0,0)^* = [I - (P_0^-)^*(P_0^+ - P_0^-)^*]|_{\operatorname{Ker}(P_0^+)^*}$. Since $P_0^+ - P_0^-$ is compact, $\dim \operatorname{Ker} N(0,0)^* < \infty$. $\square$

The assumption of Lemma 7.3 is often used in the literature on Morse theory in Hilbert spaces, in particular, for the study of Fredholm differential operator G on infinite-dimensional spaces in [2,3]. To establish a link between the current work and [2,3] assume, for a moment, that X is a Hilbert space, and (P_W, P_V) is a pair of self-adjoint projections on subspaces W and V of X, respectively. The pair (W, V) is called *commensurable* if the operator $P_W - P_V$ is compact, see [2, Chapter 2]. It can be shown that if the pair (W, V) is commensurable, then the pair $(W, V^\perp)$ is Fredholm, and

$$\operatorname{ind}(W, V^\perp) = \dim(W, V), \tag{7.2}$$

where the *relative dimension*, $\dim(W, V)$, of subspaces W and V is defined by $\dim(W, V) := \dim(W \cap V^\perp) - \dim(W^\perp \cap V)$, see [1, Section 2.2]. Here, subspaces W and V are, in general, infinite dimensional. However, if $\dim W < \infty$ and $\dim V < \infty$, then $\dim(W, V) = \dim W - \dim V$.

Example 7.4. To illustrate the simple fact that not every Fredholm pair of subspaces is commensurable, let $P_W = \frac{1}{2}\begin{bmatrix} I & I \\ I & I \end{bmatrix}$ and $P_V = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ be self-adjoint projections on the subspaces $W = \{x \oplus x : x \in H\}$ and $V = \{x \oplus 0 : x \in H\}$ of the orthogonal direct sum X of two copies of an infinite-dimensional Hilbert space H. Then $P_W - P_V$ is not compact (since it is invertible), but $W + V^\perp = X$ and $W \cap V^\perp = \{0\}$, and thus $(W, V^\perp)$ is a Fredholm pair.

If an evolution family $\{U(t,\tau)\}_{t \geqslant \tau}$ has exponential dichotomies $\{P_t^+\}_{t \geqslant 0}$ and $\{P_t^+\}_{t \leqslant 0}$ on $\mathbb{R}_+$, resp., on $\mathbb{R}_-$, then only the subspaces $\operatorname{Im} P_0^+$ (stable for $t \to \infty$) and $\operatorname{Ker} P_0^-$ (stable for $t \to -\infty$) are uniquely determined, see e.g, [Remark IV.3.4; 38, Equation (3.20)]. Thus, if X is a Hilbert space, we can assume in Propositions 7.5 and 7.15 below that P_0^+ and P_0^- are self-adjoint projections. Lemma 7.3 and formula (7.2) for $W = \operatorname{Ker} P_0^-$ and $V = \operatorname{Ker} P_0^+$ lead to the following abridged version of Theorem 1.2 that, nevertheless, covers many known results. In particular, the index formula below gives the corresponding ;formulas from [12,37], and is related to [3, Theorem B] (see also Proposition 7.15 below).

Proposition 7.5. *Suppose that an evolution family $\{U(t,\tau)\}_{t \geqslant \tau}$ on a Banach space X has exponential dichotomies $\{P_t^+\}_{t \geqslant 0}$ and $\{P_t^-\}_{t \leqslant 0}$ on $\mathbb{R}_+$ and $\mathbb{R}_-$ such that the operator $P_0^+ - P_0^-$ is compact. Then the following holds:*

(a) $\mathbf{G}$ *is Fredholm on* $L_p(\mathbb{R}; X)$, $p \in [1, \infty)$; $\operatorname{ind} \mathbf{G} = \operatorname{ind}(\operatorname{Ker} P_0^-, \operatorname{Im} P_0^+)$;
(b) *if, in addition, X is a Hilbert space and $P_0^\pm$ are self-adjoint projections, then* $\operatorname{ind} \mathbf{G} = \dim(\operatorname{Ker} P_0^-, \operatorname{Ker} P_0^+)$;
(c) *if, moreover,* $\dim \operatorname{Ker} P_0^\pm < \infty$, *then* $\operatorname{ind} \mathbf{G} = \dim \operatorname{Ker} P_0^- - \dim \operatorname{Ker} P_0^+$.

Conversely, if the operators $U(t,\tau)$, $t > \tau \in \mathbb{R}$, on a reflexive Banach space X are compact, and $\mathbf{G}$ is Fredholm, then there exist exponential dichotomies $\{P_t^+\}_{t \geqslant 0}$ and $\{P_t^-\}_{t \leqslant 0}$, and $\dim \operatorname{Ker} P_0^\pm < \infty$.

Perturbations. Consider a well-posed differential equation $u'(t) = A(t)u(t)$, $t \in \mathbb{R}$, with the propagator $\{U_A(t,\tau)\}_{t \geq \tau}$, $t, \tau \in \mathbb{R}$, and a perturbation $B : \mathbb{R} \to \mathscr{L}(X)$. We will impose the following assumptions[3] on the perturbation:

(P_1) the function $t \mapsto B(t)x$ is continuous for each $x \in X$;
(P_2) $\sup_{t \in \mathbb{R}} \|B(t)\| < \infty$;
(P_3) the perturbed equation $u'(t) = [A(t) + B(t)]u(t)$ is well posed with the propagator $\{U_{A+B}(t,\tau)\}_{t \geq \tau}$, $t, \tau \in \mathbb{R}$;
(P_4) $\lim_{|t| \to \infty} \|B(t)\| = 0$;
(P_5) $B(t)$ is a compact operator for each $t \in \mathbb{R}$.

We remark that assumption (P_3) is not trivial in view of an example due to Phillips, see, e.g., [48, Example 2.3]. Let $\mathbf{G}_A$ and $\mathbf{G}_{A+B}$ denote the generators of the evolution semigroups induced by $\{U_A(t,\tau)\}_{t \geq \tau}$ and $\{U_{A+B}(t,\tau)\}_{t \geq \tau}$, respectively. Under assumptions (P_1)–(P_3) it can be shown that $\mathbf{G}_{A+B} = \mathbf{G}_A + \mathscr{B}$, where $\mathscr{B} \in \mathscr{L}(L_p(\mathbb{R}; X))$ is defined by $(\mathscr{B}u)(t) = B(t)u(t)$, a.e. $t \in \mathbb{R}$, cf. [15, Theorem 5.24]. Obviously, $\mathscr{B}$ may not be compact. As an example, consider $\mathscr{B}$ with $B(t) = \alpha(t)B$, where $\alpha \in C_0(\mathbb{R}; \mathbb{R}), \alpha \neq 0$, and B is a compact operator such that $\sigma(B) \neq \{0\}$. Then $\sigma(\mathscr{B}) = \{\alpha(t) : t \in \mathbb{R}\} \cdot \sigma(B)$ is uncountable.

Proposition 7.6. *Suppose that B satisfies assumptions* (P_1)–(P_5). *Then* $\mathbf{G}_A$ *and* $\mathbf{G}_{A+B}$ *are Fredholm on* $L_p(\mathbb{R}; X)$, $p \in [1, \infty)$, *simultaneously, and* $\operatorname{ind} \mathbf{G}_A = \operatorname{ind} \mathbf{G}_{A+B}$.

Proof. Let D_A and D_{A+B} denote the difference operators on $\ell_p(\mathbb{Z}; X)$, $p \in [1, \infty)$, induced by the evolution families $\{U_A(t,\tau)\}_{t \geq \tau}$ and $\{U_{A+B}(t,\tau)\}_{t \geq \tau}$ using (1.5). By Theorem 1.4, we need to show that D_A and D_{A+B} are Fredholm at the same time with equal indexes. By the standard perturbation theory, the perturbed evolution family $\{U_{A+B}(t,\tau)\}_{t \geq \tau}$ satisfies a variation of constants formula for all $t \geq \tau$. This formula, in particular, implies $U_{A+B}(n+1, n)x = U_A(n+1, n)x + K_{n+1}x$, for all $x \in X$ and $n \in \mathbb{Z}$, where

$$K_{n+1}x = \int_n^{n+1} U_{A+B}(n+1, s)B(s)U_A(s, n)x \, ds.$$

Then $D_{A+B} - D_A = \mathscr{K}$, where $\mathscr{K} := \operatorname{diag}[K_n]_{n \in \mathbb{Z}} : (x_n)_{n \in \mathbb{Z}} \mapsto (K_n x_n)_{n \in \mathbb{Z}}$. Since $B(s) \to 0$ as $|s| \to \infty$ in $\mathscr{L}(X)$, and the evolution families $\{U_A(t,\tau)\}_{t \geq \tau}$ and $\{U_{A+B}(t,\tau)\}_{t \geq \tau}$ are exponentially bounded, we have $\lim_{|n| \to \infty} K_n = 0$ in $\mathscr{L}(X)$. Also, since operators $B(s)$, $s \in \mathbb{R}$, are compact and the functions $f_n(\cdot) = U_{A+B}(n + 1, \cdot)B(\cdot)U_A(\cdot, n)$ are strongly continuous on $[n, n+1], n \in \mathbb{Z}$, we conclude that K_n is compact in X for each $n \in \mathbb{Z}$, see, e.g. [19, p. 525]. Thus, $\mathscr{K}$ is compact in $\ell_p(\mathbb{Z}; X)$ as a limit in $\mathscr{L}(\ell_p(\mathbb{Z}; X))$ of a sequence of compact operators. $\square$

[3] Apparently, the assumption that $B(t)$, $t \in \mathbb{R}$, are *bounded* operators could be relaxed to include wider classes of perturbations, cf. [15, Section 5.2.2], but we will not pursue this here.

Asymptotically constant coefficients: Let A be the generator of a strongly continuous semigroup $\{e^{tA}\}_{t\geqslant 0}$ on X. The evolution family corresponding to the equation $u'(t) = Au(t)$ is given by $U(t,\tau) = e^{(t-\tau)A}$ for $t \geqslant \tau$ in $\mathbb{R}$. Recall that a semigroup $\{e^{tA}\}_{t\geqslant 0}$ is called *hyperbolic* on X if there exists a projection P_A such that $e^{tA}P_A = P_A e^{tA}$, $t\geqslant 0$, and that $\|e^{tA}|_{\mathrm{Im}\,P_A}\| \leqslant Me^{-\alpha t}$, $t\geqslant 0$, $\alpha>0$, and the semigroup $\{e^{tA}|_{\mathrm{Ker}\,P_A}\}_{t\geqslant 0}$ extends to a strongly continuous *group* $\{e^{tA}|_{\mathrm{Ker}\,P_A}\}_{t\in\mathbb{R}}$ on Ker P_A such that $\|e^{tA}|_{\mathrm{Ker}\,P_A}\| \leqslant Me^{\alpha t}$, $t\leqslant 0$, see, e.g., [15, p. 28]. The semigroup $\{e^{tA}\}_{t\geqslant 0}$ is hyperbolic if and only if $\sigma(e^{tA})\cap\mathbb{T} = \emptyset$ for some (and hence for all) $t>0$. Then P_A is the spectral (Riesz) projection for $\{e^{tA}\}_{t\geqslant 0}$ such that $\sigma(e^{tA}|_{\mathrm{Im}\,P_A}) = \sigma(e^{tA})\cap\{\lambda\in\mathbb{C} : |\lambda|<1\}$, see [15, Lemma 2.15].

Lemma 7.7. *Assume that for some $b\geqslant 0$ the evolution family $\{e^{(t-\tau)A}\}_{t\geqslant\tau}$, $t,\tau\in\mathbb{R}$, has either an exponential dichotomy $\{P_t^+\}_{t\geqslant b}$ on $[b,+\infty)$, or an exponential dichotomy $\{P_t^-\}_{t\leqslant -b}$ on $(-\infty,-b]$. Then the semigroup $\{e^{tA}\}_{t\geqslant 0}$ is hyperbolic on X.*

Proof. We will prove that $\sigma(e^A)\cap\mathbb{T} = \emptyset$ provided there is a dichotomy $\{P_t^+\}_{t\geqslant b}$. First, we claim that $\|(I-e^A)x\|\geqslant c\|x\|$ for some $c>0$ and all $x\in X$. By Lemma 5.4, for some $c>0$ we have $\|D_b^+\mathbf{x}\|_{\ell_p}\geqslant c\|\mathbf{x}\|_{\ell_p}$ for all $\mathbf{x}\in\ell_p(\mathbb{Z}\cap[b+1,\infty);X)$. For each $x\in X$ and $\gamma>0$ define $\mathbf{x} = (e^{-\gamma n}x)_{n\geqslant b+1}$. Then $D_b^+\mathbf{x} = (e^{-\gamma(b+1)}x, (e^{-\gamma(b+2)} - e^A e^{-\gamma(b+1)})x, \ldots)$. A calculation shows that

$$\begin{aligned}\|D_b^+\mathbf{x}\|_{\ell_p}^p &= e^{-\gamma p(b+1)}\{\|x\|^p + \|(e^{-\gamma} - e^A)x\|^p/(1-e^{-\gamma p})\}\\ &\geqslant c^p\|\mathbf{x}\|_{\ell_p}^p = c^p e^{-\gamma p(b+1)}\|x\|^p/(1-e^{-\gamma p}).\end{aligned}$$

Thus, $(1-e^{-\gamma p})\|x\|^p + \|(e^{-\gamma}-e^A)x\|^p \geqslant c^p\|x\|^p$, and letting $\gamma\to 0$ the claim is proved. Rescaling $A\mapsto A-i\beta$, $\beta\in\mathbb{R}$, shows that $\|(\lambda - e^A)x\|\geqslant c\|x\|$ for all $x\in X$ and $\lambda\in\mathbb{T}$. To finish the proof of the lemma, it suffices to show that $\sigma_p((e^A)^*)\cap\mathbb{T} = \emptyset$ for the point spectrum $\sigma_p(\cdot)$. Arguing by contradiction and using the Spectral Mapping Theorem for the point spectrum ([19, Section IV.3.b], and also see [19, Section IV.2.18]), suppose that $A^*\xi = i\beta\xi$ for some $\beta\in\mathbb{R}$ and $\xi\in X^*$. Then $(e^{sA})^*\xi = e^{i\beta s}\xi$ for all $s\geqslant 0$. Using the dichotomy $\{P_t^+\}_{t\geqslant b}$ and passing to the adjoints, for all $t\geqslant b$ we have $(P_b^+)^*(e^{(t-b)A})^* = (e^{(t-b)A})^*(P_t^+)^*$, and the dichotomy estimates $\|(e^{(t-b)A})^*|_{\mathrm{Im}(P_t^+)^*}\|\leqslant Me^{-\alpha(t-b)}$, $\|((e^{(t-b)A})^*|_{\mathrm{Ker}(P_t^+)^*})^{-1}\|\leqslant Me^{-\alpha(t-b)}$. Denote $\xi_t = e^{-i\beta(t-b)}(e^{(t-b)A})^*\xi$, $t\geqslant b$. Identity $(e^{(t-b)A})^*\xi = e^{i\beta(t-b)}\xi$ implies that $\xi = (P_b^+)^*\xi_t + (I-(P_b^+)^*)\xi_t$ for all $t\geqslant b$. By the stable dichotomy estimate $\|(P_b^+)^*\xi_t\| = \|(e^{(t-b)A})^*(P_t^+)^*\xi\|\leqslant Me^{-\alpha(t-b)}\sup_{t\geqslant b}\|(P_t^+)^*\|\|\xi\|$, and we have $\lim_{t\to\infty}(I-(P_b^+)^*)\xi_t = \lim_{t\to\infty}[\xi - (P_b^+)^*\xi_t] = \xi\in\mathrm{Ker}(P_b^+)^*$ since $(I-(P_b^+)^*)\xi_t\in\mathrm{Ker}(P_b^+)^*$. By the unstable dichotomy estimate,

$$\|\xi\| = \|(I-(P_b^+)^*)\xi_t\| = \|(e^{(t-b)A})^*(I-(P_t^+)^*)\xi\|\geqslant M^{-1}e^{\alpha(t-b)}\|(I-(P_t^+)^*)\xi\|,$$

and so $\lim_{t\to\infty}(I-(P_t^+)^*)\xi=0$. Using the decomposition $\xi=(P_t^+)^*\xi+(I-(P_t^+)^*)\xi$, we have $\xi=\lim_{t\to\infty}(P_t^+)^*\xi$. Remark that $\operatorname{Im} P_t^+=\operatorname{Im} P_b^+$ for all $t\geqslant b$. Indeed, using the dichotomy $\{P_t^+\}_{t\geqslant b}$, for each $t\geqslant b$ we infer:

$$\begin{aligned}\operatorname{Im} P_t^+ &= \{x\in X : \|e^{A(s-t)}x\|\leqslant Me^{-\alpha(s-t)}\|x\| \text{ for all } s\geqslant t\}\\ &= \{x\in X : \|e^{A\tau}x\|\leqslant Me^{-\alpha\tau}\|x\| \text{ for all } \tau\geqslant 0\} = \operatorname{Im} P_b^+.\end{aligned}$$

Since $\operatorname{Im}(P_t^+)^* = (\operatorname{Im} P_t^+)^*$, for all $t\geqslant b$ we thus have $\operatorname{Im}(P_t^+)^*=\operatorname{Im}(P_b^+)^*$. Therefore, $(P_t^+)^*\xi\in\operatorname{Im}(P_b^+)^*$ implies $\xi=\lim_{t\to\infty}(P_t^+)^*\xi\in\operatorname{Im}(P_b^+)^*$, and so $\xi=0$ since we have proved that $\xi\in\operatorname{Ker}(P_b^+)^*\cap\operatorname{Im}(P_b^+)^*$. Dichotomy $\{P_t^-\}_{t\leqslant -b}$ is considered similarly. $\square$

Corollary 7.8. *Let A be the generator of a strongly continuous semigroup on a reflexive Banach space X. Then the following assertions are equivalent*:

(1) $\mathbf{G}_A$ *is Fredholm on* $L_p(\mathbb{R};X)$, $p\in[1,\infty)$;
(2) $\mathbf{G}_A$ *is invertible on* $L_p(\mathbb{R};X)$, $p\in[1,\infty)$;
(3) $\sigma(e^{tA})\cap\mathbb{T}=\emptyset$ *for all* $t>0$.

Proof. Equivalence (2) $\Leftrightarrow$ (3) is contained in [15, Theorem 3.13]. To prove (1) $\Rightarrow$ (3), apply Theorem 1.1. By this theorem, (1) implies the existence of an exponential dichotomy $\{P_t^+\}_{t\geqslant b}$ on $[b,\infty)$ for the evolution family $\{e^{(t-\tau)A}\}_{t\geqslant\tau}$. By Lemma 7.7 the semigroup $\{e^{tA}\}_{t\geqslant 0}$ is hyperbolic. $\square$

Next, consider a perturbed differential equation $u'(t)=[A+B(t)]u(t)$, $t\in\mathbb{R}$. If assumption (P_4) holds then this equation is asymptotically autonomous (for a recent work on asymptotically autonomous parabolic equations see also [10,20,47,49]).

Lemma 7.9. *Suppose that assumptions* (P_1)–(P_4) *hold. Assume that for some $b\geqslant 0$ the evolution family $\{U_{A+B}(t,\tau)\}_{t\geqslant\tau}$ for $u'(t)=[A+B(t)]u(t)$, $t\in\mathbb{R}$, has either an exponential dichotomy $\{P_t^+\}_{t\geqslant b}$ on $[b,\infty)$, or an exponential dichotomy $\{P_t^-\}_{t\leqslant -b}$ on $(-\infty,-b]$. Then the semigroup $\{e^{tA}\}_{t\geqslant 0}$ is hyperbolic.*

Proof. Suppose that the evolution family $\{U_{A+B}(t,\tau)\}_{t\geqslant\tau}$ has an exponential dichotomy $\{P_t^+\}_{t\geqslant b}$ on $[b,\infty)$ with the dichotomy constants α, M. Since $B(t)\to 0$ in $\mathscr{L}(X)$ as $t\to\infty$ by assumption (P_4), for each $\varepsilon\in(0,\alpha(2M)^{-1})$ there exists a $T=T(\varepsilon)\geqslant b$ such that $\sup\{\|B(t)\| : t\geqslant T\}<\varepsilon$. For $t\in\mathbb{R}$ we set $\tilde{P}_t=P_t^+$ if $t\geqslant T$ and $\tilde{P}_t=P_T^+$ if $t<T$. Also, we define a strongly continuous exponentially bounded evolution family $\{\tilde{U}_{A+B}(t,\tau)\}_{t\geqslant\tau}$, $t,\tau\in\mathbb{R}$, a continuation of

$\{U_{A+B}(t,\tau)\}_{t\geqslant\tau\geqslant T}$, by

$$\tilde{U}_{A+B}(t,\tau)=\begin{cases} U_{A+B}(t,\tau) & \text{for } t\geqslant\tau\geqslant T,\\ U_{A+B}(t,T)e^{\alpha(T-\tau)(I-2P_T^+)} & \text{for } t\geqslant T\geqslant\tau,\\ e^{\alpha(t-\tau)(I-2P_T^+)} & \text{for } T\geqslant t\geqslant\tau,\end{cases} \tag{7.3}$$

cf. [12, p. 109]. Since $e^{\alpha(t-\tau)(I-2P_T^+)}=e^{-\alpha(t-\tau)}P_T^+ + e^{\alpha(t-\tau)}(I-P_T^+)$, it is easy to check that $\{\tilde{P}_t\}_{t\in\mathbb{R}}$ is an exponential dichotomy for $\{\tilde{U}_{A+B}(t,\tau)\}_{t\geqslant\tau}$ on $\mathbb{R}$ with the same dichotomy constants α, M. By Chicone and Latushkin [15, Theorem 3.13], the generator $\tilde{\mathbf{G}}_{A+B}$ of the evolution semigroup on $L_p(\mathbb{R};X)$ induced by $\{\tilde{U}_{A+B}(t,\tau)\}_{t\geqslant\tau}$ is invertible, and, moreover, $\|(\tilde{\mathbf{G}}_{A+B})^{-1}\|_{\mathscr{L}(L_p(\mathbb{R};X))}\leqslant 2M\alpha^{-1}$, see, e.g. [15, p. 105]. Extend the evolution family $\{e^{(t-\tau)A}\}_{t\geqslant\tau\geqslant T}$ as follows:

$$\tilde{U}_{A}(t,\tau)=\begin{cases} e^{(t-\tau)A} & \text{for } t\geqslant\tau\geqslant T,\\ e^{A(t-T)}e^{\alpha(T-\tau)(I-2P_T^+)} & \text{for } t\geqslant T\geqslant\tau,\\ e^{\alpha(t-\tau)(I-2P_T^+)} & \text{for } T\geqslant t\geqslant\tau.\end{cases} \tag{7.4}$$

Define $\tilde{B}:\mathbb{R}\to\mathscr{L}(X)$ by setting $\tilde{B}(t)=B(t)$ for $t\geqslant T$ and $\tilde{B}(t)=0$ for $t<T$, and define $\tilde{\mathscr{B}}\in\mathscr{L}(L_p(\mathbb{R};X))$ by $\tilde{\mathscr{B}}u(t)=\tilde{B}(t)u(t),\ t\in\mathbb{R}$. Then $\tilde{\mathbf{G}}_{A+B}=\tilde{\mathbf{G}}_A+\tilde{\mathscr{B}}$, where $\tilde{\mathbf{G}}_A$ is the generator of the evolution semigroup on $L_p(\mathbb{R};X)$ induced by the evolution family $\{\tilde{U}_A(t,\tau)\}_{t\geqslant\tau}$. By the choice of T,

$$\|\tilde{\mathscr{B}}\|_{\mathscr{L}(L_p(\mathbb{R};X))}=\sup_{t\geqslant T}\|B(t)\|<\varepsilon<\alpha(2M)^{-1}\leqslant(\|(\tilde{\mathbf{G}}_{A+B})^{-1}\|_{\mathscr{L}(L_p(\mathbb{R};X))})^{-1}.$$

Thus, $\tilde{\mathbf{G}}_A$ is invertible on $L_p(\mathbb{R}_+;X)$ since $\tilde{\mathbf{G}}_{A+B}$ is invertible. By Chicone and Latushkin [15, Theorem 3.13], the evolution family $\{\tilde{U}_A(t,\tau)\}_{t\geqslant\tau}$ has an exponential dichotomy on $\mathbb{R}$, hence, on $[T,\infty)$, and thus $\{e^{tA}\}_{t\geqslant 0}$ is hyperbolic by Lemma 7.7 with $b=T$. The case of exponential dichotomy on $(-\infty,b]$ is considered similarly. □

Proposition 7.10. *Assume that A is the generator of a strongly continuous semigroup on a reflexive Banach space X, and assumptions* (P$_1$)–(P$_5$) *hold for a perturbation $B:\mathbb{R}\to\mathscr{L}(X)$. Then $\mathbf{G}_{A+B}$ is Fredholm on $L_p(\mathbb{R};X)$, $p\in[1,\infty)$, if and only if the semigroup $\{e^{tA}\}_{t\geqslant 0}$ is hyperbolic. Moreover*, $\operatorname{ind}\mathbf{G}_{A+B}=0$.

Proof. If $\mathbf{G}_{A+B}$ is Fredholm, then $\{U_{A+B}(t,\tau)\}_{t\geqslant\tau}$ has an exponential dichotomy on $\mathbb{R}_+$ by Theorem 1.2. By Lemma 7.9, $\{e^{tA}\}_{t\geqslant 0}$ is hyperbolic. Conversely, if $\{e^{tA}\}_{t\geqslant 0}$ is hyperbolic then $\mathbf{G}_A$ is invertible on L_p, see Corollary 7.8. By Proposition 7.6 $\mathbf{G}_{A+B}$ is Fredholm and $\operatorname{ind}\mathbf{G}_{A+B}=0$. □

There is an alternative proof of Lemma 7.9, appropriate for $C_0(\mathbb{R};X)$, that uses difference operators, cf. the proof of Proposition 7.6. This proof is based on the fact that if D_{A+B} and D_A are the difference operators (1.5) induced by the evolution families defined in (7.3) and (7.4), respectively, then $\|D_{A+B} - D_A\|$ is small provided $\|\tilde{\mathscr{B}}\|$ is small. Also, because of Lemma 7.9, assumption (P_5) on B was used only in the proof of the "only if" part of Proposition 7.10. Thus, for any $B \in C_0(\mathbb{R}; \mathscr{L}(X))$, if $\mathbf{G}_{A+B}$ is Fredholm, then $\{e^{tA}\}_{t \geqslant 0}$ is hyperbolic.

Asymptotically piecewise constant coefficients: Let A_+ and A_- be the generators of strongly continuous semigroups $\{e^{tA_+}\}_{t\geqslant 0}$ and $\{e^{tA_-}\}_{t\geqslant 0}$ on X, respectively. Assume that $\operatorname{dom} A_+ = \operatorname{dom} A_-$, and let

$$A_0(t) = A_+ \text{ for } t \geqslant 0 \quad \text{and} \quad A_0(t) = A_- \text{ for } t < 0. \tag{7.5}$$

Then the differential equation $u'(t) = A_0(t)u(t),\ t \in \mathbb{R}$, is well-posed in the W_p^1-sense with a propagator $\{U(t,\tau)\}_{t\geqslant\tau},\ t,\tau \in \mathbb{R}$, defined as follows:

$$U(t,\tau) = \begin{cases} e^{(t-\tau)A_+} & \text{for } t \geqslant \tau \geqslant 0, \\ e^{tA_+}e^{-\tau A_-} & \text{for } t \geqslant 0 \geqslant \tau, \\ e^{(t-\tau)A_-} & \text{for } 0 \geqslant t \geqslant \tau. \end{cases} \tag{7.6}$$

The invertibility of $\mathbf{G}_{A_0}$ with bounded operators $A_\pm$ has been studied in [14].

Proposition 7.11. *Let A_0 be defined by* (7.5), *where* $\operatorname{dom} A_+ = \operatorname{dom} A_-$. *The operator $\mathbf{G}_{A_0}$ is Fredholm on $L_p(\mathbb{R}; X)$, $p \in [1, \infty)$, if and only if*

(1) *the semigroups $\{e^{tA_+}\}_{t\geqslant 0}$ and $\{e^{tA_-}\}_{t\geqslant 0}$ are hyperbolic on X with the spectral projections P_{A_+} and P_{A_-}, respectively*;
(2) *the node operator $N(0,0) = (I - P_{A_+})|_{\operatorname{Ker} P_{A_-}} : \operatorname{Ker} P_{A_-} \to \operatorname{Ker} P_{A_+}$ is Fredholm.*

Moreover, $\dim \operatorname{Ker} \mathbf{G}_{A_0} = \dim \operatorname{Ker} N(0,0)$, $\operatorname{codim} \operatorname{Im} \mathbf{G}_{A_0} = \operatorname{codim} \operatorname{Im} N(0,0)$, *and* $\operatorname{ind} \mathbf{G}_{A_0} = \operatorname{ind} N(0,0)$.

Proof. If (1) and (2) hold then $\mathbf{G}_{A_0}$ is Fredholm and the index formula is valid by the "if" part of Theorem 1.2. If $\mathbf{G}_{A_0}$ is Fredholm, then by the "only if" part of Theorem 1.2, there exist dichotomies $\{P_t^+\}_{t\geqslant 0}$ and $\{P_t^-\}_{t\leqslant 0}$ for the evolution family $\{U(t,\tau)\}_{t\geqslant\tau}$ defined in (7.6). By Lemma 7.7, the semigroups $\{e^{tA_\pm}\}_{t\geqslant 0}$ are hyperbolic, and we may set $P_t^+ = P_{A_+}$ and $P_t^- = P_{A_-}$. This proves (1). Assertion (2) holds by the implication (1.3) $\Rightarrow$ (ii′) in Theorem 1.2, and Lemma 5.1. $\square$

Next, consider $A(t) = A_0(t) + B(t)$ with B satisfying assumptions (P_1)–(P_3), and let $\mathbf{G}_{A_0+B}$ and $\mathbf{G}_{A_0}$ denote the generators of the evolution semigroups induced by the propagators of the differential equations $u'(t) = [A_0(t) + B(t)]u(t)$ and $u'(t) = A_0(t)u(t)$, respectively. Recall that if $\sigma(A) \cap i\mathbb{R} = \emptyset$ then P_A denotes the spectral projection such that $\sigma(A|_{\operatorname{Im} P_A}) = \sigma(A) \cap \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < 0\}$.

Proposition 7.12. *Assume that A_+ and A_-,* $\operatorname{dom} A_+ = \operatorname{dom} A_-$, *are the generators of strongly continuous semigroups on a reflexive Banach space X, and B satisfies assumptions* (P_1)–(P_5). *The operator $\mathbf{G}_{A_0+B}$ is Fredholm if and only if the semigroups $\{e^{tA_+}\}_{t\geqslant 0}$ and $\{e^{tA_-}\}_{t\geqslant 0}$ are hyperbolic with the spectral projections P_{A_+} and P_{A_-}, and the pair of subspaces* $(\operatorname{Ker} P_{A_-}, \operatorname{Im} P_{A_+})$ *is Fredholm. Moreover,* $\operatorname{ind} \mathbf{G}_{A_0+B} = \operatorname{ind}(\operatorname{Ker} P_{A_-}, \operatorname{Im} P_{A_+})$.

Proof. By Proposition 7.6, $\mathbf{G}_{A_0+B}$ and $\mathbf{G}_{A_0}$ are Fredholm at the same time, and their indexes are equal. The rest follows from Proposition 7.11 and Lemma 5.1. □

Corollary 7.13. *Let X be a separable Hilbert space. Assume that A_+ and A_- are selfadjoint operators with compact resolvent, and* $\operatorname{dom} A_+ = \operatorname{dom} A_-$. *Let A_0 be defined as in* (7.5). *Suppose that $B : \mathbb{R} \to \mathscr{L}(X)$ satisfies assumptions* (P_1)–(P_5), *and that $B(t)$ is a selfadjoint operator for each $t \in \mathbb{R}$. Then $\mathbf{G}_{A_0+B}$ is Fredholm if and only if A_+ and A_- are invertible. Moreover,* $\operatorname{ind} \mathbf{G}_{A_0+B}$ *is equal to the spectral flow for the family $A(t) = A_0(t) + B(t),\ t \in \mathbb{R}$.*

Recall, that the spectral flow for the family $\{A(t)\}_{t\in\mathbb{R}}$ of selfadjoint operators with compact resolvent represents the net change in the number of negative eigenvalues of $A(t)$ as t changes from $-\infty$ to $+\infty$, see e.g. [41] or [33, Section 8.16]. In the situation described in Corollary 7.13 we thus *define* the spectral flow as $\dim \operatorname{Ker} P_{A_-} - \dim \operatorname{Ker} P_{A_+}$, cf. [20]. Note that $A(t)$ has compact resolvent for all $t \in \mathbb{R}$.

Proof. By the spectral mapping theorem $\sigma(e^{tA})\backslash\{0\} = \exp t\sigma(A),\ t > 0$, for selfadjoint operators [19, Theorem IV.3.10], the operator $A_\pm$ is invertible if and only if the semigroup $\{e^{tA_\pm}\}_{t\geqslant 0}$ is hyperbolic. Since A_+ and A_- have compact resolvents, $\operatorname{Ker} P_{A-}$ and $\operatorname{Ker} P_{A_+}$ are finite dimensional, and $P_{A_+} - P_{A_-}$ is compact. Thus, subspaces $\operatorname{Ker} P_{A_-}$ and $\operatorname{Ker} P_{A_+}$ are commensurable, and, by Lemma 7.3, the node operator $N(0,0)$ is Fredholm. So, by Lemma 5.1 the pair of subspaces $(\operatorname{Ker} P_{A_-}, \operatorname{Im} P_{A_+})$ is Fredholm. Using formula (7.2) for $W = \operatorname{Ker} P_{A_-}$ and $V = (\operatorname{Im} P_{A_+})^\perp$, we conclude that $\operatorname{ind}(\operatorname{Ker} P_{A_-}, \operatorname{Im} P_{A_+}) = \dim \operatorname{Ker} P_{A_-} - \dim \operatorname{Ker} P_{A_+}$. An application of Proposition 7.12 concludes the proof. □

Bounded coefficients: Assume that (7.1) holds, and recall that $\mathbf{G}_A = G_A$. Let $\{U(t,\tau)\}_{t,\tau\in\mathbb{R}}$ denote the propagator for $u'(t) = A(t)u(t),\ t \in \mathbb{R}$. If $\{U(t,\tau)\}_{t,\tau\in\mathbb{R}}$ has exponential dichotomies $\{P_t^+\}_{t\geqslant 0}$ and $\{P_t^-\}_{t\leqslant 0}$ on $\mathbb{R}_+$ and $\mathbb{R}_-$, then the stable, W_A^s, and unstable, W_A^u, subspaces for A can be described as follows:

$$W_A^s = \left\{x \in X : \lim_{t\to\infty} U(t,0)x = 0\right\} = \operatorname{Im} P_0^+,$$

$$W_A^u = \left\{x \in X : \lim_{t\to-\infty} U(t,0)x = 0\right\} = \operatorname{Ker} P_0^-.$$

Proposition 7.14. *Assume that A satisfies* (7.1). *Then the operator G_A is Fredholm if and only if the following holds*: (a) *There exist exponential dichotomies $\{P_t^+\}_{t\geqslant 0}$ and $\{P_t^-\}_{t\leqslant 0}$ on $\mathbb{R}_+$ and $\mathbb{R}_-$ for $\{U(t,\tau)\}_{t,\tau\in\mathbb{R}}$; and* (b) *The pair of subspaces (W_A^s, W_A^u) is Fredholm. Moreover*, $\operatorname{ind} G = \operatorname{ind}(W_A^s, W_A^u)$.

This follows from Theorem 1.2. Further, if the limits $A_+ = \lim_{t\to\infty} A(t)$ and $A_- = \lim_{t\to-\infty} A(t)$ exist in $\mathscr{L}(X)$, and $\sigma(A_\pm)\cap i\mathbb{R} = \emptyset$, then the operator family $\{A(t)\}_{t\in\mathbb{R}}$ is called an *asymptotically hyperbolic* path; see, e.g., [3]. Under the additional assumption that $\{A(t)\}_{t\in\mathbb{R}}$ is asymptotically hyperbolic, Proposition 7.14 has been proved in [3, Theorem D]. Our results show, however, that if the limits A_+ and A_- exist and the operator G_A is Fredholm, then $\sigma(A_\pm)\cap i\mathbb{R} = \emptyset$. Indeed, since G_A is Fredholm, Theorem 1.1 implies the existence of dichotomies $\{P_t^+\}_{t\geqslant b}$ and $\{P_t^-\}_{t\leqslant a}$ for some $a\leqslant b$. Using the assumption that $A_\pm = \lim_{t\to\pm\infty} A(t)$ exist in $\mathscr{L}(X)$, this, in turn, implies that $\sigma(e^{tA_\pm})\cap\mathbb{T} = \emptyset$, $t>0$, see Lemma 7.9. Further, for $A_\pm\in\mathscr{L}(X)$ define A_0 as in (7.5), and consider a compact-valued perturbation $B:\mathbb{R}\to\mathscr{L}(X)$ that satisfies assumptions (P_1)–(P_5). Proposition 7.12 and formula (7.2) give the following improvement of [3, Theorem B], where the "if" part of Proposition 7.15 has been proved.

Proposition 7.15. *If $A(t) = A_0(t) + B(t)$, $t\in\mathbb{R}$, where A_0 is given by* (7.5) *with $A_\pm\in\mathscr{L}(X)$, and B takes compact values and vanishes at $\pm\infty$, then G_A is Fredholm on $L_p(\mathbb{R};X)$, $p\in[1,\infty)$, if and only if $\sigma(A_\pm)\cap i\mathbb{R}=\emptyset$ and the pair of the spectral subspaces* $(\operatorname{Im} P_{A_+}, \operatorname{Ker} P_{A_-})$ *for A_+ and A_- is Fredholm. Moreover*, $\operatorname{ind} G_A = \operatorname{ind}(\operatorname{Ker} P_{A_-}, \operatorname{Im} P_{A_+})$. *If X is a Hilbert space and, in addition, $A_+ - A_-$ is a compact operator, and $P_{A_\pm}$ are self-adjoint projections, then* $\operatorname{ind} G_A = \dim(\operatorname{Ker} P_{A_-}, \operatorname{Ker} P_{A_+}) = \dim(\operatorname{Im} P_{A_+}, \operatorname{Im} P_{A_-})$.

Connections to Morse Theory: A need to study Fredholm properties and the index of the operator G naturally arises in infinite-dimensional Morse theory, see [1,2] and the literature therein. If $X=\mathbb{R}^d$ and v is a (heteroclinic) solution of the equation $v'(t) = f(v(t))$ connecting two hyperbolic stagnation points, $x_- = \lim_{t\to-\infty} v(t)$ and $x_+ = \lim_{t\to\infty} v(t)$, then the linearization along v gives rise to the operator $Gu = -u' + A(t)u$, where $A(t) = Df(v(t))$, $t\in\mathbb{R}$, and Df is the differential. If f is a gradient vector field, that is, $f=-DF$ for a Morse functional $F:X\to\mathbb{R}$ (such that $D^2F(x)$ is hyperbolic at all critical points x of F), then $A(\pm\infty) = -D^2F(x_\pm)$, and the number $\dim\operatorname{Ker} P_{-D^2F(x_\pm)} = \dim\operatorname{Ker} P_{A(\pm\infty)}$ is called the Morse index of the critical point $x_\pm$. It is well-known that $\operatorname{ind} G = \dim\operatorname{Ker} P_{A(-\infty)} - \dim\operatorname{Ker} P_{A(+\infty)}$, see, e.g., [41, Theorem 2.1]. If X is an infinite-dimensional Hilbert space then Morse functionals of particular interest are of the form $F(x) = \frac{1}{2}\langle Ax, x\rangle + b(x)$ since they appear in the study of Hamiltonian systems, wave equations, and some elliptic systems, see [1,2]. Here A is a self-adjoint operator and the Hessian $D^2F(x) = A + D^2b(x)$, where $D^2b(x)$ is a compact operator on X for each $x\in X$. If, as above, v

is a heteroclinic trajectory connecting (hyperbolic) critical points, then the linearization along v gives the operator $Gu = -u' + A(t)u$, where $A(t) = A + B(t)$, $B(t) = D^2b(v(t))$, $t \in \overline{\mathbb{R}}$. In the infinite-dimensional situation just outlined, the Morse theory has been developed in [2]. Note, that the results of the current section (see Proposition 7.12 and Corollary 7.13) show that the hyperbolicity of the operators $D^2F(x_\pm)$ is, in fact, necessary for the operator G to be Fredholm. Moreover, it appears that Theorem 1.2 is applicable for more general Morse functionals. In this case, the exponential dichotomies on $\mathbb{R}_\pm$ in this theorem seem to be a correct generalization of the asymptotic hyperbolicity.

Travelling waves: Applications of the finite-dimensional Dichotomy Theorem in the theory of travelling waves are important and well-understood, see [45] and the literature therein. We briefly sketch a simple generalization of the setup in [45], suitable for applications of the infinite-dimensional version of this theorem given in the current paper (cf. [38,46, pp. 89–91]). Let Y be a Banach space, $\mathcal{N} : Y \to Y$ be a differentiable non-linear map, $\mathbf{p}(\cdot)$ be a polynomial with constant coefficients, $u : \mathbb{R}_+ \times \mathbb{R} \to Y$. Consider a non-linear equation

$$\partial_t u = \mathbf{p}(\partial_x)u + \mathcal{N}(u), \quad t \in \mathbb{R}_+, \ x \in \mathbb{R}. \tag{7.7}$$

A typical situation occurs when $u = u(t, x, y)$, $y \in \mathbb{R}^d$, and $Y = L_2(\mathbb{R}^d)$, so that $u(t, \cdot, \cdot) \in L_2(\mathbb{R} \times \mathbb{R}^d) = L_2(\mathbb{R}; L_2(\mathbb{R}^d))$ and $u(t, x, \cdot) \in L_2(\mathbb{R}^d)$. In our general setting, passing to the moving frame $\xi = x - ct$, $c \neq 0$, $v(t, \xi) = u(t, \xi + ct)$, $\xi \in \mathbb{R}$, we have that u satisfies (7.7) if and only if v satisfies

$$\partial_t v = \mathbf{p}(\partial_\xi)v + c\partial_\xi v + \mathcal{N}(v), \quad t \in \mathbb{R}_+, \quad \xi \in \mathbb{R}. \tag{7.8}$$

A function $\mathbf{q} = \mathbf{q}_c(\xi)$, $\mathbf{q} : \mathbb{R} \to Y$, is called a *travelling wave* for (7.7) if $\mathbf{q}$ is a t-independent solution of (7.8), that is, if $\mathbf{p}(\partial_\xi)\mathbf{q} + c\partial_\xi\mathbf{q} + \mathcal{N}(\mathbf{q}) = 0$. Assume that the latter (non-linear) equation has a solution. A linearization of (7.8) about $\mathbf{q}$ gives rise to an operator

$$Lw := \mathbf{p}(\partial_\xi)w + c\partial_\xi w + D\mathcal{N}(\mathbf{q}(\xi))w, \quad w = w(\xi) \in Y, \ \xi \in \mathbb{R}. \tag{7.9}$$

In a "general" semilinear case we might have $\mathcal{N}(u) = Nu + F(u)$, where N is any generator of a strongly continuous semigroup on Y. If, in addition, $DF(0) = 0$, $\mathbf{q}(\xi) \to 0$ as $|\xi| \to \infty$, and for each $\xi \in \mathbb{R}$ the operator $B(\xi) = DF(\mathbf{q}(\xi))$ is a compact operator on Y, then our perturbation results are applicable. Finally, we note that the eigenvalue problem $Lw = \lambda w$ for L in (7.9) is a higher order non-autonomous ordinary differential equation in Y and, as such, could be rewritten as a first order equation $u'(\xi) = A(\xi)u(\xi)$, where $A(\xi), \xi \in \mathbb{R}$, depends on λ and, generally, is an unbounded differential operator on a suitable Banach space $X = Y \oplus \cdots \oplus Y$. Thus, the spectrum of L is related to the Fredholm properties of the operator $\mathbf{G}_A$ induced by A which are described in the current paper.

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